Final Physics 204 May 8, 2007

Do this exam by yourself. Each of the problems is worth 10 points. For partial credit you must show your work. If figures are required, please draw them big and clear. No calculators or books are needed or permitted.

1. A bar of soap (mass m) is at rest on a frictionless rectangular tray that rests on a horizontal table. At $t = 0$, one edge of the tray starts being raised, so that the tray pivots about the opposite edge with a constant angular velocity ω, and the soap begins sliding downhill toward the edge of the tray which is still touching the table.

(a) Write the Lagrangian as a function of x, where x is the soap's distance from the opposite edge touching the table's surface. [3 points]

(b) Show that the equation of motion for the soap has the form $\ddot{x} - \omega^2 x = -g \sin \omega t$. [2 points]

(c) Solve the above equation for $x(t)$, given that $x(0) = x_0$. (The complementary solution is well known. For the particular solution, try $x_p(t) = A \sin \omega t$ and then solve for A.) [5 points]

2. A pointlike body of mass m can move without friction on a vertical cone (with vertex pointing down) with an opening of 2α.

(a) Write the Lagrangian of this system using the cylindrical coordinates θ and r. [5 points]

(b) Which coordinate is ignorable? What does that imply? [1 point]

(c) Integrate the equations of motion to obtain the total energy. Several terms could be grouped together to form the 'effective potential', $V_{\text{eff}}(r)$. What is $V_{\text{eff}}(r)$ in this case? [3 points]

(d) Obtain $t(r)$ as an integral of $dr/\dot{r}$. (But do not actually do the integral.) [1 points]

3. An archetypical harmonic oscillation consists of a body of mass m which can slide on a horizontal surface without friction, and which is attached to a wall with a spring of length ℓ_0 and spring constant k. Let x be the extension of the spring from its equillibrium length.

(a) Write the Lagrangian for the case of the massless spring, obtain the equation of motion and the angular frequency ω. [2 points]

(b) Let us now consider a more realistic case in which the spring is *not* massless, but rather has a mass M. Assuming that the spring is uniform and stretches uniformly, show that it's kinetic energy is $T_{\text{spring}} = M\dot{x}^2/6$. [5 points]

(c) Write down the Lagrangian of the system with a massive string, keeping in mind that the potential energy is still $U(x) = kx^2/2$. [2 points]

(d) Show that the system with the massive spring still behaves like a harmonic oscillator, but with the modified angular frequency $\omega = \sqrt{k(m + M/3)}$. [1 point]

4. The axes of two cylinders of equal radii R and masses m are connected with two massless rods of length L, in order to form a simplistic 'cart'. However, one cylinder is homogenous and the other has thin wall. This 'cart' is then placed on an incline plane making an angle α with respect to the horizontal.

(a) Write the Lagrangian for this system using θ (the rotation angle of the disks) as the generalized variable. [2 points]

(b) Write the Lagrangian for the system of two independent cylinders of the same type – e.g. of the same system but without the rods. In addition to θ, use the distance between the axes of the two cylinders as the second generalized coordinate. [2 points]

(c) For the case (b) to reduce to (a), which constrainst needs to be satisfied? [1 point]

(d) Finally, using the method of Lagrangian undetermined multipliers, determine the force of the tension in the rods. [5 points]

5. A cone of mass m, height h and radius at the base R, is attached at the vertex to a bar of height equal to R above a plane, so that the axis of the cone is parallel to the plane. The plane is inclined at the angle α with respect to the horizontal. The base of the cone is touching the incline plane and can roll without slipping on its surface. The vertex is attached in such a way to allow the rotation of the cone around both its axis of symmetry as well as the axis perpendicular to the inclined plane (which passes through the bar holding the vertex).

(a) Calculate the center of mass and the inertia tensor of the cone around its vertex. [4 points]

(b) What is the kinetic energy of the rolling cone? [3 points]

(c) Write the Lagrangian. [1 point]

(d) Obtain the angular frequency of small oscillations. [2 points]

6. (bonus) A thin cylinder of radius R and mass m is spinning with a high angular velocity ω_0. At the initial moment, the cylinder is placed on a horizontal surface. (The coefficient of kinetic friction between the cylinder and the surface is μ.)

(a) Describe qualitatively the motion of the cylinder. [2 points]

(b) Derive the velocity of the axis of the cylinder, $v(t)$ and its angular velocity $\omega(t)$ as a function of time. [3 points]

(c) Determine the time when the slipping stops. [3 points]

(d) What's the fraction of the initial kinetic energy of the cylinder that is dissipated as heat? [2 points]

(Note: this problem involves dissipative forces so use Newton's laws.)