171.106 Final Exam, spring 2011

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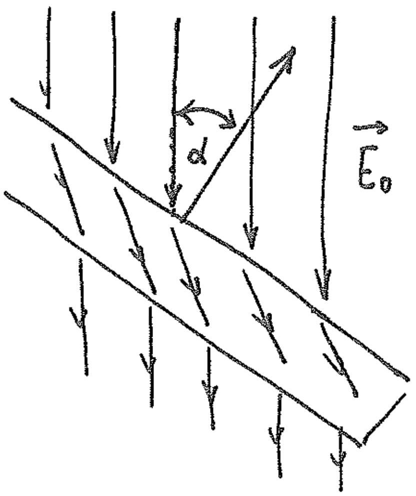
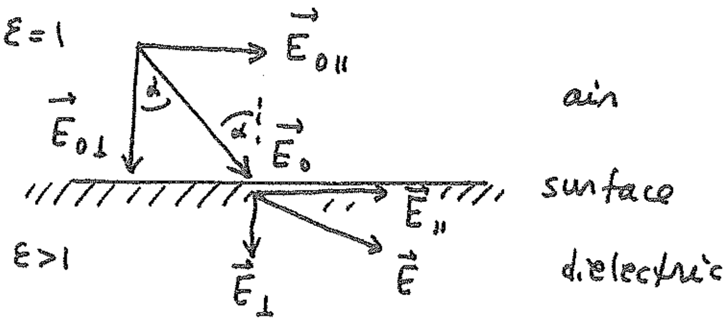




At the surface, the perpendicular component $\vec{E}_\perp$ gets weaker by ε.

$$\left.\begin{aligned} E_{0\parallel} &= E_0 \sin\alpha \\ E_{0\perp} &= E_0 \cos\alpha \end{aligned}\right\} \text{air}$$

$E_\parallel = E_{0\parallel}$ ← not affected by surf. charge due to polarized medium

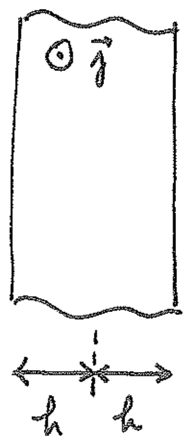
$$E_\perp = \frac{E_{0\perp}}{\varepsilon}$$

$E = \sqrt{E_\parallel^2 + E_\perp^2}$ ← magnitude inside dielectric

$$E = \sqrt{E_{0\parallel}^2 + \left(\frac{E_{0\perp}}{\varepsilon}\right)^2} = \sqrt{E_0^2 \sin^2\alpha + \frac{1}{\varepsilon^2} E_0^2 \cos^2\alpha}$$

$$\boxed{E = E_0 \sqrt{\sin^2\alpha + \frac{1}{\varepsilon^2}\cos^2\alpha}}$$

$\boxed{2}$



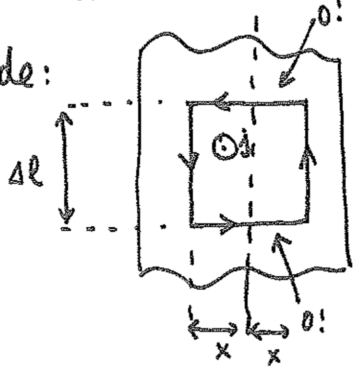
We use Ampere's Law

$$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi k}{c}\sum_i I_i$$

We form Ampere's loop as a square, symmetric about the midplane:

a)

inside:



$$\oint \vec{B}\cdot d\vec{l} = \int_{up} \vec{B}\cdot d\vec{l} + \underbrace{\cancel{\int \vec{B}\cdot d\vec{l}}}_{top \to 0} + \int_{down} \vec{B}\cdot d\vec{l} + \underbrace{\cancel{\int \vec{B}\cdot d\vec{l}}}_{bottom \to 0}$$

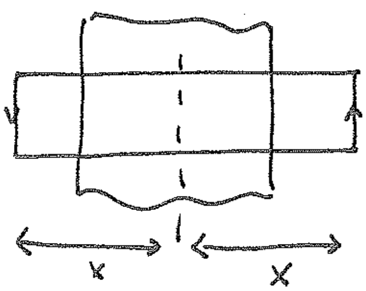
$$= B_{up}(x)\cdot\Delta l + B_{down}(x)\cdot\Delta l$$

But, due to symmetry, $B_{up} = B_{down} = B(x)$

$$\oint \vec{B}\cdot d\vec{l} = \cancel{2}B(x)\cancel{\Delta l} = \frac{4\pi k}{c}\sum_i I_i = \frac{4\pi k}{c}\cdot j\,\cancel{2}x\cancel{\Delta l}$$

$$\boxed{B(x) = \frac{4\pi k j}{c}x \text{ inside}}$$

outside:

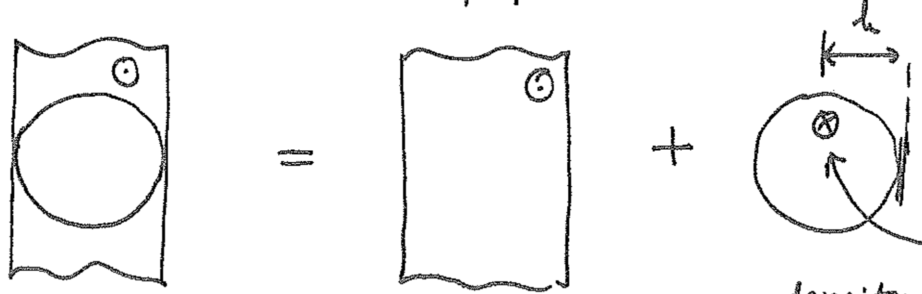


The difference now is that the current enclosed is only up to $x = h$:

$$\cancel{2}B(x)\cancel{\Delta l} = \frac{4\pi k}{c}j\,\cancel{2}h\cancel{\Delta l}$$

$$\boxed{B(x) = \frac{4\pi k}{c}jh = \text{const}}$$

$\boxed{\text{2 cont'd}}$ b) We represent the cylindrical hole as a cylinder with current flowing backwards – into the paper.



but note opposite current (turns + into –)

The field of the cylinder with current density j is

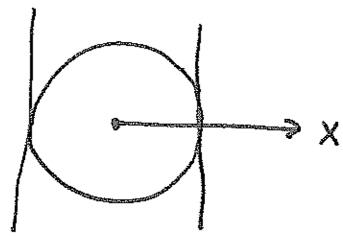
<u>inside:</u> $\oint \vec{B}\cdot d\vec{l} = B(r)\cdot 2\pi r = \frac{4\pi k}{c}\pi r^2 j$

$$B(r) = \frac{2\pi k j}{c}\cdot r$$

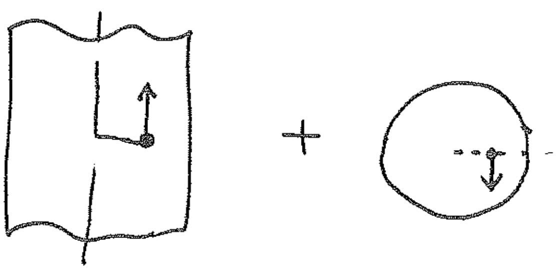
<u>outside:</u> $\oint \vec{B}\cdot d\vec{l} = B(r)\cdot 2\pi r = \frac{4\pi k}{c}\pi h^2 j$

$$B(r) = \frac{2\pi k j}{c}\,\frac{h^2}{r}$$

Putting everything together, field along x:



$\vec{B}(x) = \vec{B}_{sheet} + \vec{B}_{cyl.}$ ← but, along x axis, both are colinear, so magnitudes add (subtract)



$\boxed{\text{2 cont'd}}$ $\underline{x < h:}$

$$B(x) = \frac{4\pi k j}{c} x + \frac{2\pi k(-j)}{c} x = \boxed{\frac{2\pi k j}{c} \cdot x}$$

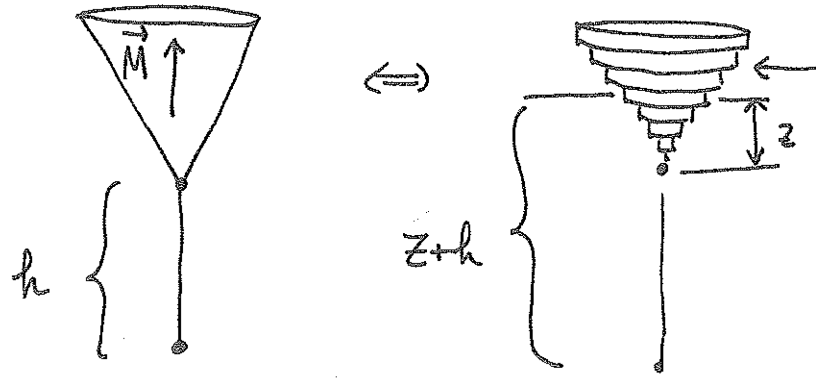
$\underline{x > h:}$

$$B(x) = \frac{4\pi k}{c} jh - \frac{2\pi k j}{c} \frac{h^2}{x} =$$

$$\boxed{= \frac{2 \cdot \pi k j h}{c} \left[2 - \frac{h}{x}\right]}$$

Note that as $x \to +\infty$, the effect of the hole becomes negligible, and the field approaches that of an infinite sheet.

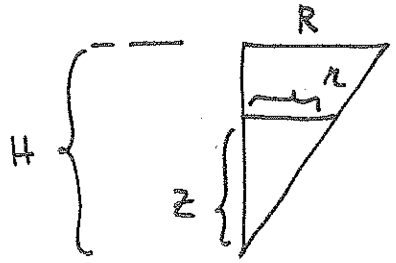
$\boxed{3}$ We model the magnetized cone with a stack of thin disks.



each disk has a dipole moment of

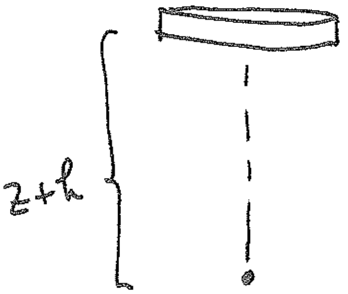
$$d\vec{m} = \vec{M} \cdot dV = \vec{M}\, a\, dz$$

where $a = r(z)^2 \pi =$

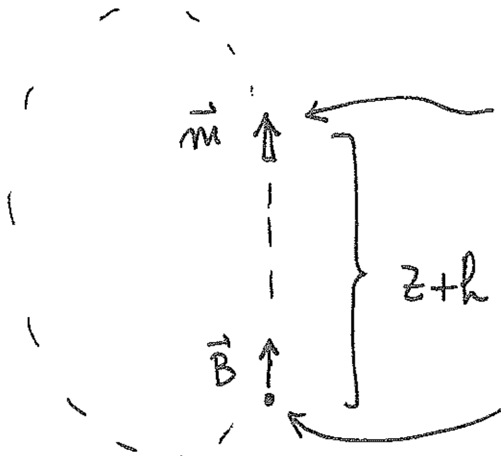


$$\frac{r}{z} = \frac{R}{H} \Rightarrow r(z) = \frac{R}{H} z$$

$$\Rightarrow a(z) = \left(\frac{R z}{H}\right)^2 \pi = \frac{R^2 \pi}{H^2} z^2$$



← contribution of one disk is approximated by a dipole moment at distance $z+h$



dipole moment $dm = \dfrac{M R^2 \pi}{H^2} z^2 dz$

The field at the point will be oriented along the z axis.

$$B_z = \frac{m\,(3\cos^2\theta - 1)}{r^3} = \frac{3m}{(z+h)^3} \quad \left(\text{since } \theta = 0 \text{ and } r = z+h\right)$$

3 cont'd

$$dB_z(h) = \frac{3\,dm}{(z+h)^3}$$

Total B field is $\vec{B} = \hat{z}\int dB_z$, and is oriented along z axis:

$$\boxed{\vec{B} = \hat{z}\int \frac{3\,dm}{(z+h)^3} = \hat{z}\,\frac{3MR^2\pi}{H^2}\int_0^H \frac{z^2\,dz}{(z+h)^3}}$$

(see next page for the integral)

$$\boxed{\vec{B} = \hat{z}\;\frac{3MR^2\pi}{H^2}\left[\ln\left(1+\frac{H}{h}\right) + \frac{2hH}{(h+H)^2}\right]}$$

$\boxed{\text{3 cont'd}}$ The integral:

$$\int_0^H \frac{z^2\,dz}{(z+h)^3} \xlongequal[\substack{du = dz \\ z = u-h}]{u = z+h} \int_h^{h+H} \frac{(u-h)^2\,du}{u^3} = \int_h^{h+H} \frac{u^2-2uh+h^2}{u^3}\,du =$$

$$= \int_h^{h+H} \frac{du}{u} - 2h\int_h^{h+H} \frac{du}{u^2} + h^2 \int_h^{h+H} \frac{du}{u^3} =$$

$$= \ln u \Big|_h^{h+H} - 2h(-1)\frac{1}{u}\Big|_h^{h+H} + h^2(-2)\frac{1}{u^2}\Big|_h^{h+H} =$$

$$= \ln\frac{h+H}{h} + 2h\left[\frac{1}{h+H} - \frac{1}{h}\right] - 2h^2\left[\frac{1}{(h+H)^2} - \frac{1}{h^2}\right] =$$

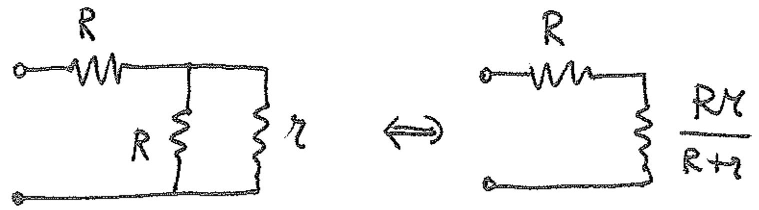
$$= \ln\frac{h+H}{h} + 2\cancel{h}\left[\frac{\cancel{h} - \cancel{h} - H}{(h+H)\cancel{h}}\right] - 2\cancel{h^2}\left[\frac{h^2 - (h+H)^2}{(h+H)^2\cancel{h^2}}\right]$$

$$= \ln\frac{h+H}{h} - \frac{2H}{h+H} + \frac{2}{(h+H)^2}\left[(h+H+h)(\cancel{h}+H-\cancel{h})\right]$$

$$= \ln\left(1+\frac{H}{h}\right) + \frac{2H}{h+H}\left[-1 + \frac{H+2h}{H+h}\right]$$

$$= \ln\left(1+\frac{H}{h}\right) + \frac{2hH}{(h+H)^2}$$

$\boxed{4}$ a)



$$R_{eq} = R + \frac{Rr}{R+r}$$

b) $R_{eq} = r = R + \frac{Rr}{R+r} \Longleftrightarrow r - R = \frac{Rr}{R+r}$

$$(r-R)(r+R) = Rr$$

$$r^2 - R^2 = Rr$$

$$r^2 - Rr - R^2 = 0$$

Quadratic equation, with $a=1$, $b=-R$, $c=-R^2$

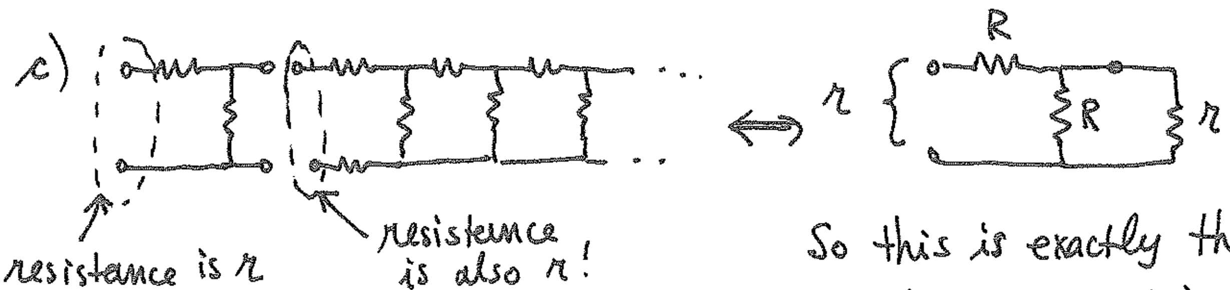
$$r_{1,2} = \frac{R \pm \sqrt{R^2 - 4\cdot(-R^2)}}{2} = \frac{1}{2}\left[R \pm \sqrt{5}\cdot R\right]$$

$$r_{1,2} = \frac{R}{2}\left[1 \pm \sqrt{5}\right]$$

Since $\sqrt{5} > 1$, the root with "$-$" is unphysical, and we do not consider it.

$$\Rightarrow \boxed{r = \frac{R}{2}\left(1+\sqrt{5}\right)}$$

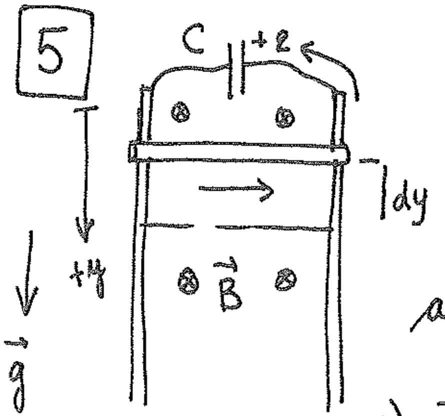
c)



So this is exactly the same situation as in (b)

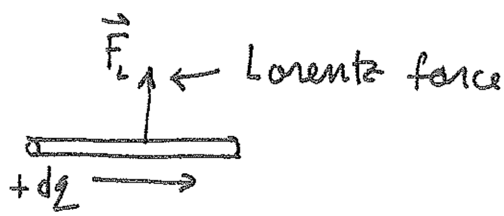
$$\boxed{R_{eq} = r = \frac{R}{2}\left(1+\sqrt{5}\right)} \quad !$$

5



As the bar is falling down, the flux through the circuit increases, generating an e.m.f.

a) $$\mathcal{E} = -\frac{1}{c}\frac{d\phi}{dt} = -\frac{1}{c}\,\ell B\frac{dy}{dt} = -\frac{\ell B v_y}{c}$$



b) The induced current is flowing to the right, and thus the Lorentz force on the charges in the bar acts up – it slows the falling of the bar.

$$F_L = \frac{1}{c} I B \ell$$ where I is the current.

c) There are no resistances in the circuit, so the voltage on the capacitor will always have to be the same as the generated e.m.f:

$V = \mathcal{E}$, but $V = \frac{Q}{C}$, where Q is the charge on the capacitor.

$$\frac{Q}{C} = \frac{1}{c} B \ell v_y$$

The current can be obtained from the rate of increase of the charge on the capacitor: $I = \frac{dQ}{dt}$

$$\Rightarrow \quad I = \frac{dQ}{dt} = \frac{d}{dt}\left[C \cdot \frac{B\ell v_y}{c}\right] = C \cdot \frac{B\ell}{c}\frac{dv_y}{dt} = C \cdot \frac{B\ell}{c} a_y$$

So the current is directly proportional to the acceleration of the bar.

5 cont'd

d) 2^{nd} Newton's law:

$$ma_y = m\ddot{y} = mg - F_L = mg - \frac{B\ell}{c} \cdot I$$

But $I = C \frac{B\ell}{c} a_y$

$$\Rightarrow \quad ma_y = mg - \frac{B\ell}{c} \cdot C \frac{B\ell}{c} a_y = mg - C\frac{B^2\ell^2}{c^2} a_y$$

Solving for a_y:

$$\left[m + C\frac{B^2\ell^2}{c^2}\right] a_y = mg$$

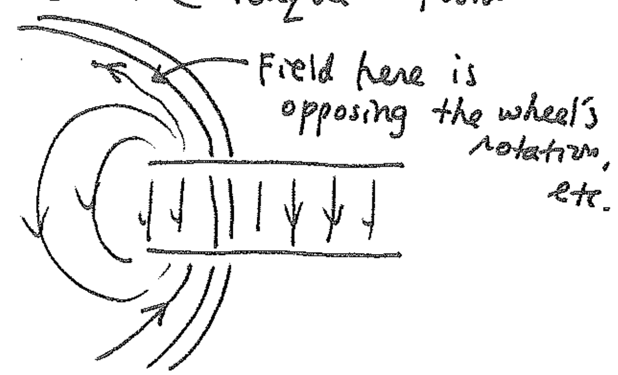
$$\Rightarrow \boxed{a_y = \frac{mg}{m + C\frac{B^2\ell^2}{c^2}}} = \text{const}!$$

The differential equation is simple, and the solution is $v_y = a_y t$ since at $t=0$, $y=0$ and $v_y = 0$

$$\Rightarrow \boxed{v_y(t) = \frac{mg}{m + C\frac{B^2\ell^2}{c^2}} \cdot t}$$

6

a) The problem with the inventor's argument is that there are fringe fields "leaking" out of the capacitor. They are weak, but the ring is also large, and thus, integrated over the rest of the ring (outside the capacitor), the total torque is sufficient to cancel the torque from within the capacitor:

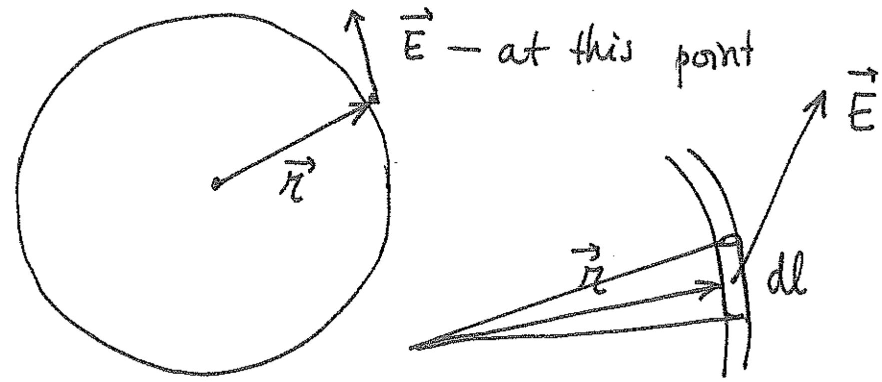


b) Any static charge distribution must obey $\oint \vec{E} \cdot d\vec{l} = 0$ over any closed loop. (This comes from the fact that the Coulomb's force is central, and from application of the principle of superposition.)

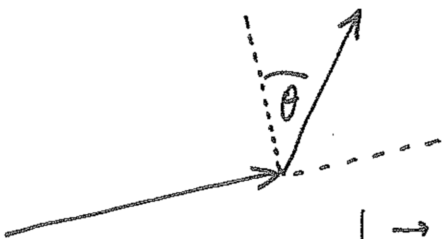
Thus, the integral $\oint \vec{E} \cdot d\vec{l}$ over the ring as contour must be $= 0$ as well!

We now use this to compute the total torque about the center of the ring.

6 cont'd



Consider dl somewhere on the ring. Force on it is $(\lambda dl)\vec{E}$, where λ is a constant linear charge density of the ring. The torque is $d\vec{\tau} = \vec{r} \times (\lambda dl \vec{E})$



Let θ be the angle between $d\vec{l}$ and $\vec{E}$. Since $\vec{r} \perp d\vec{l}$, then

$$|\vec{r} \times \vec{E}| = rE\cos\theta$$

$$\Rightarrow \quad d\tau = r\lambda dl E\cos\theta = r\lambda(dl E\cos\theta) = r\lambda(d\vec{l}\cdot\vec{E})$$

Since all $d\vec{\tau}$ are collinear, we add them in the integral

$$\tau = \oint d\tau = \oint r\lambda dl E\cos\theta = \oint r\lambda \vec{E}\cdot d\vec{l} =$$

$$= r\lambda \oint \vec{E}\cdot d\vec{l} = 0 \quad \text{since } \oint \vec{E}\cdot d\vec{l} = 0$$

Q.E.D.