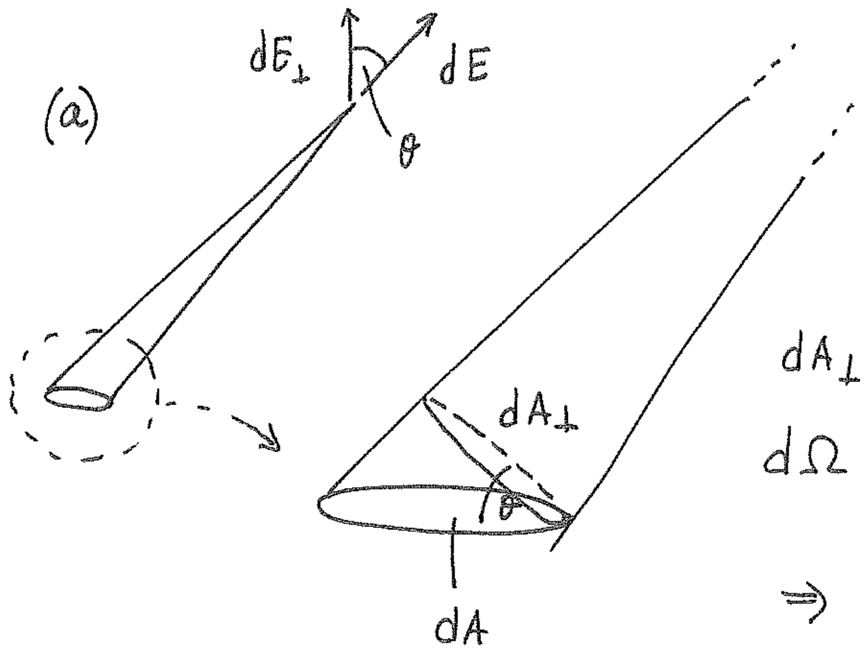
171. 106
The First Midterm,
March 2011

SOLUTIONS

1 (a)



$dA_\perp = dA\cos\theta$

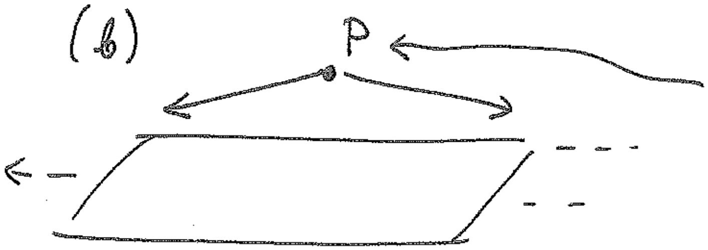
$d\Omega = \frac{dA_\perp}{r^2} = \frac{dA\cos\theta}{r^2}$

$\Rightarrow dA = \frac{r^2 d\Omega}{\cos\theta}$

$$dE_\perp = dE\cos\theta = k\frac{dq}{r^2}\cos\theta = k\frac{\sigma\, dA}{r^2}\cos\theta$$

$$\Rightarrow dE_\perp = k\sigma\, d\Omega \Rightarrow E_T = \int dE_\perp = k\sigma\int d\Omega = \underline{\underline{k\sigma\Omega}}$$

(b)



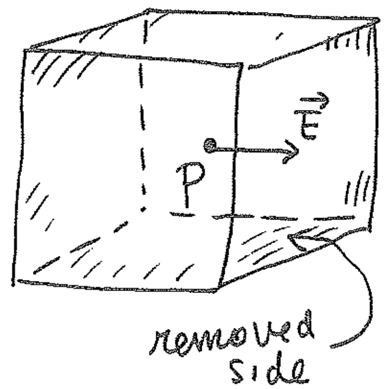
Point P "sees" the whole infinite plane as half of the whole solid angle $\Rightarrow \Omega = \frac{4\pi}{2} = 2\pi$

$$\Rightarrow E \equiv E_\perp = k\sigma 2\pi = 2\pi k\sigma$$

← same as with Gauss's law

1 cont'd

(c)



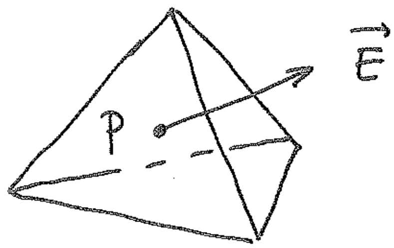
The solid angle of one side is $\Omega = \frac{4\pi}{6} = \frac{2}{3}\pi$

$\vec{E}$ is oriented toward the missing side. With all six sides, total $\vec{E}$ at P would be 0.

With the side removed, the field is opposite as it would be from the side alone:

$$\Rightarrow \quad E = E_{\perp}^{side} = k\sigma \cdot \frac{2\pi}{3}$$

(d)



Ditto for the regular tetrahedron:

$$\Omega = \frac{4\pi}{4} = \pi$$

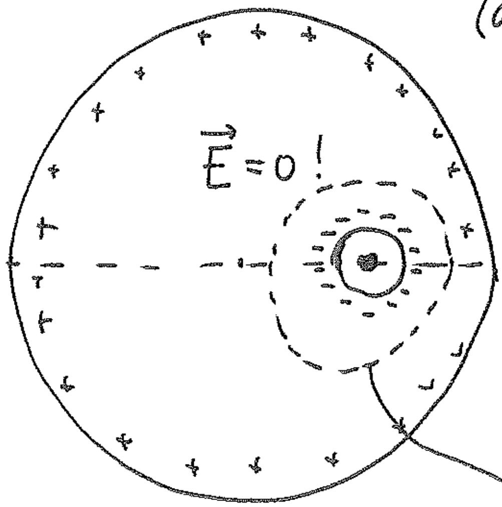
Due to symmetry, with all four sides present, the field is zero.

$\Rightarrow$ Thus, without one side, Σfield = field from one side but with negative sign.

$$\Rightarrow \quad E = k\sigma\pi$$

2

(a)



Assuming that $Q>0$, the walls of the cavity are populated with negative charges which must sum to $\underline{-Q}$ because of Gauss's law:

surface S, $\int_S \vec{E}\cdot d\vec{s} = 0$ since $\vec{E}=0$

$\Rightarrow 4\pi k \Sigma Q = 0$

(b) As the field in conductor is zero everywhere, the surface doesn't "know" anything about what is happening in the cavity $\rightarrow$ in fact, it does not know that it may even exist!

$\Rightarrow$ Thus, in the absence of external fields, the charge will distribute uniformly around the surface of the sphere.

(c) If the sphere (without Q in the cavity) is neutral, that means that there's charge Q on the surface, distributed uniformly

$$\Rightarrow \boxed{\sigma = \frac{Q}{4\pi R^2}}$$

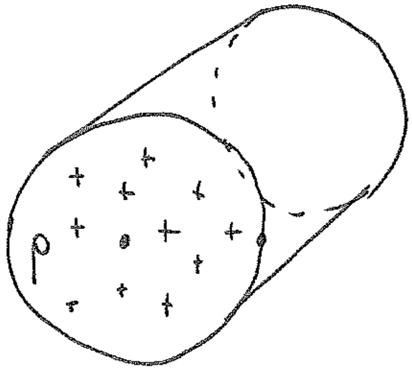
2 cont'd

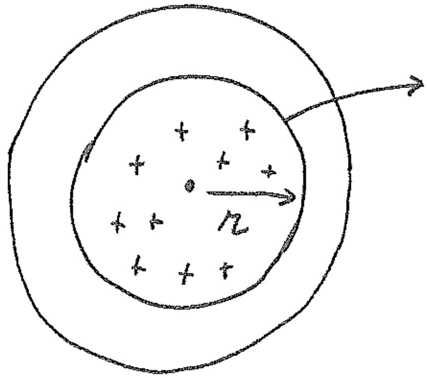
(d) As discussed in (b), the electric field outside the sphere does not know anything about the cavity $\Rightarrow$ the location and shape of the cavity are irrelevant!

If the total charge of the sphere without Q is q, then the total charge of the surface of the sphere is $Q+q$, and the field at L is:

$$E(L) = k\frac{\sum Q}{L^2} = k\frac{Q+q}{L^2}$$

$\boxed{3}$



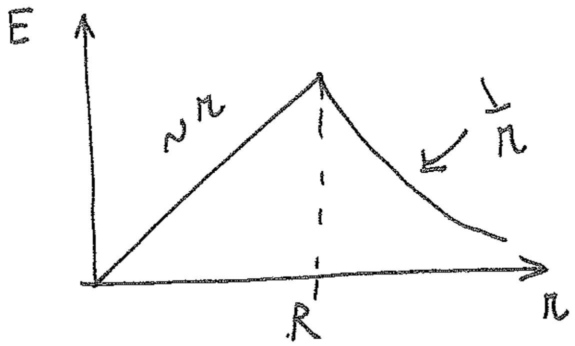


surface for Gauss's Law is a cylinder with radius r and length L

(a)

Gauss's Law: $$E(r)\cdot \underbrace{2\pi r\cdot L}_{\substack{\text{surface}\\ \text{(bases}\to 0)}} = 4\pi k\cdot \underbrace{\rho\cdot \underbrace{\pi r^2\cdot L}_{\text{volume}}}_{\text{charge}}$$

$$\Rightarrow \quad E(r) = 4\pi k\rho\,\frac{r}{2} = \underline{2\pi k\rho\cdot r} \quad \leftarrow \text{inside.}$$



Outside $$E(r)\cdot 2\cancel{\pi} r\cdot \cancel{L} = 4\pi k\rho\,\cancel{\pi}R^2\cancel{L}$$

$$E(r) = \frac{2\pi k\rho R^2}{r}$$

3 cont'd

(b) We obtain the potential inside the rod by "integrating backwards" from $r = R$ where we know $\varphi(R) = \varphi_0$.

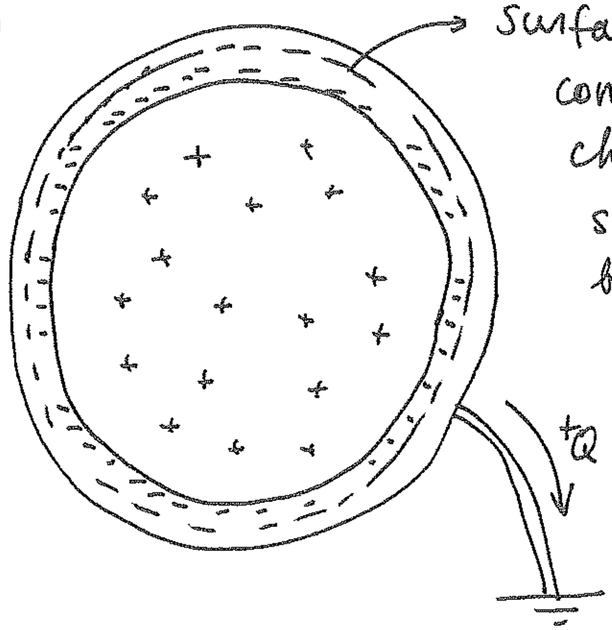
$$\vec{E} = -\nabla\varphi \qquad \text{Here} \qquad E(r) = -\frac{d\varphi}{dr}$$

$$\Rightarrow \underbrace{\varphi(R)}_{=\varphi_0} - \varphi(r) = -\int_r^R E(r)\,dr = -2\pi k\rho \int_r^R r\,dr$$

$$\Rightarrow \varphi(r) = \varphi(r) + 2\pi k\rho \int_r^R r^2\,dr = \varphi_0 + \cancel{2}\pi k\rho \frac{1}{\cancel{2}}(R^2 - r^2)$$

$$\Rightarrow \boxed{\varphi(r) = \varphi_0 + \pi k\rho\,[R^2 - r^2]}$$

(c)



Surface for Gauss's Law: $\vec{E} = 0$ in conductor $\rightarrow$ the amount of charge in inner surface of the cylinder must balance ρ inside:

$$\sum_i Q_i = 0 \iff \pi R^2 \rho \cdot L + Q = 0$$

$$\Rightarrow \boxed{Q = -\pi R^2 \rho L}$$