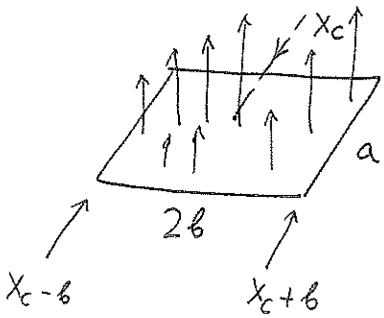
171.106 : 2^{nd} midterm 2011 – SOLUTIONS

1

a)



$$\Phi = \int_{\square} \vec{B}\cdot d\vec{a} = \int_{x_c-b}^{x_c+b} B_0 e^{\alpha x}\cdot a \cdot dx =$$

$$= a B_0 \frac{1}{\alpha} e^{\alpha x}\Big|_{x_c-b}^{x_c+b} =$$

$$= \frac{aB_0}{\alpha}\left[e^{\alpha(x_c+b)} - e^{\alpha(x_c-b)}\right] =$$

$$= \frac{aB_0}{\alpha} e^{\alpha x_c} \underbrace{\left[e^{\alpha b} - e^{-\alpha b}\right]}_{2\sinh(\alpha b)} = \frac{2aB_0}{\alpha} e^{\alpha x_c} \underbrace{\sinh(\alpha b)}_{\text{indeed, depends } \sim \sinh(\alpha b)}$$

b) frame is moving along +x with v, so $x_c = vt$

$$\Rightarrow \Phi(t) = \frac{2aB_0}{\alpha} e^{\alpha v t} \sinh(\alpha b)$$

$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{1}{c}\alpha v \frac{2aB_0}{\alpha} e^{\overbrace{\alpha vt}^{= x_c}} \sinh(\alpha b) =$$

$$= \boxed{2aB_0 v \, e^{\alpha x_c} \sinh(\alpha b)}$$

c) $R = [2a + 2\cdot 2b]\rho = 2\rho(a+2b)$

Power:

$$P(x_c) = \frac{\mathcal{E}(x_c)^2}{R} = \frac{2\cancel{4}\, a^2 B_0^2 v^2 e^{2\alpha x_c} \sinh^2(\alpha b)}{\cancel{2}\,\rho(a+2b)}$$

$$\boxed{P = \frac{2a^2B_0^2v^2 e^{2\alpha x_c}\sinh^2(\alpha b)}{\rho(a+2b)}}$$

$\boxed{1}$ cont'd

d) Total amount of heat

$$x_c = v\,t$$
$$dx_c = v\,dt \rightarrow dt = \frac{dx_c}{v}$$

$$W = \int P(t)\,dt = \int_{-\infty}^{0} P(x_c)\,\frac{dx_c}{v}$$

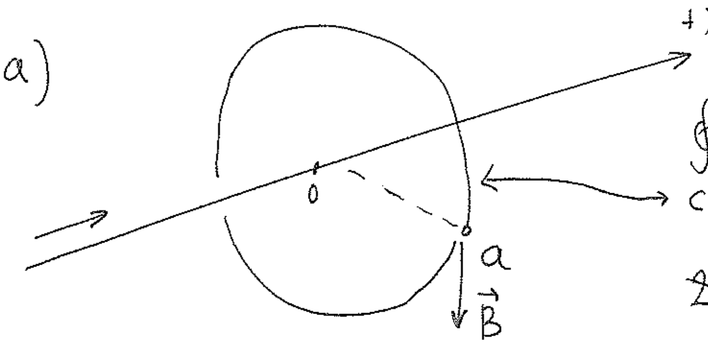
$$= \frac{2a^2 B_0^2 v^{\cancel{2}} \sinh^2(\alpha b)}{\rho(a+2b)} \int_{-\infty}^{0} e^{2\alpha x_c}\,\frac{dx_c}{\cancel{v}} =$$

$$= \frac{\cancel{2}a^2 B_0^2 v \sinh^2(\alpha b)}{\rho(a+2b)}\,\frac{1}{\cancel{2}\alpha}\Big[\underbrace{e^{2\alpha\cdot 0}}_{=1} - \underbrace{e^{-2\alpha\cdot\infty}}_{\to 0}\Big]$$

$$\boxed{W = \frac{a^2 B_0^2 v \sinh^2(\alpha b)}{\alpha\rho(a+2b)}}$$

$\boxed{2}$

a)



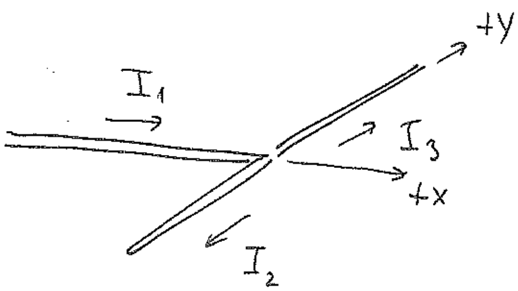
$$\oint \vec{B} \cdot d\vec{l} = \frac{4\pi k I}{c}$$

$$\cancel{2\pi} a \cdot B(a) = \frac{\cancel{4\pi}^{\,2} k I}{c}$$

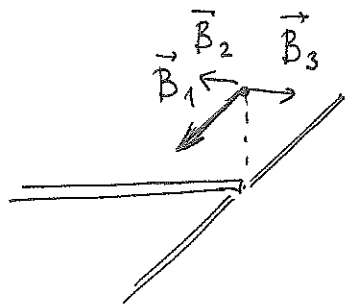
$$\Rightarrow \quad B(a) = \frac{2kI}{c} \cdot \frac{1}{a}$$

b) Note that one half of this comes from $[-\infty, 0]$ and the other half from $[0, +\infty]$. They add, since for every piece of the wire the contribution $d\vec{B} \sim \frac{d\vec{l} \times \hat{r}}{r^2}$ is oriented in the same direction - down.

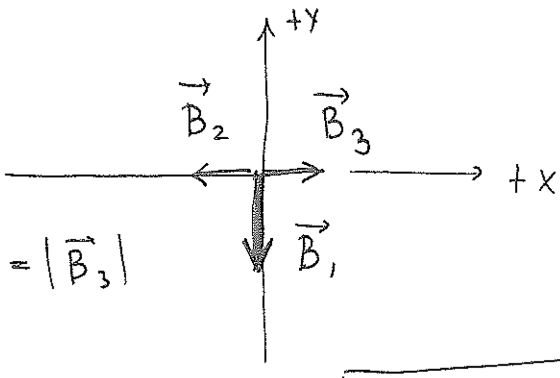
$\Rightarrow$ For semi-infinite wire, $B_{1/2} = \frac{kI}{c} \frac{1}{a}$



They create



Seen from above:



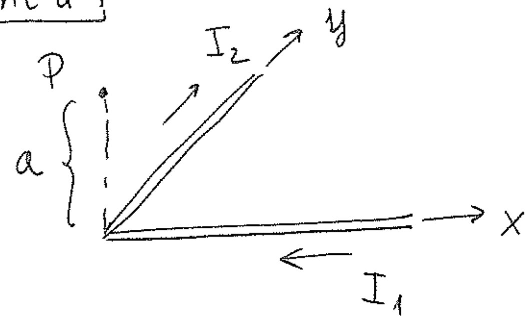
Since $I_2 = I_3 \Rightarrow |\vec{B}_2| = |\vec{B}_3|$

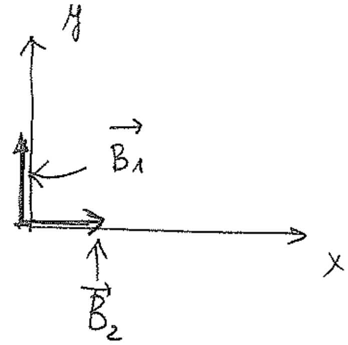
and $\vec{B}_2 + \vec{B}_3 = 0$

$$\Rightarrow \vec{B} \equiv \vec{B}_3 = \frac{k(2I)}{ca} \cdot (-\hat{y})$$

$$\boxed{\vec{B} = -\hat{y} \, \frac{2kI}{ca}}$$

$\boxed{2}$ cont'd



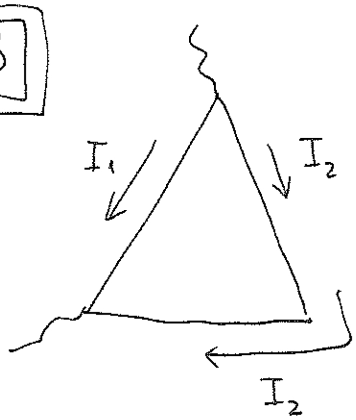


Here $|\vec{B}_1| = |\vec{B}_2|$ because $I_1 = I_2 = I$

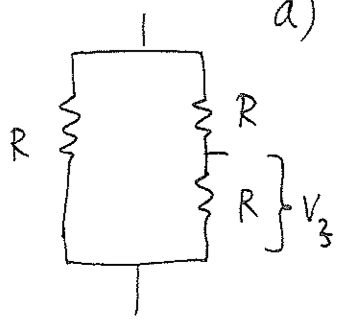
$$\Rightarrow \boxed{\vec{B} = \vec{B}_1 + \vec{B}_2} = |\vec{B}_1| \cdot \hat{y} + |\vec{B}_2| \cdot \hat{x} =$$

$$= \boxed{\frac{kI}{ca} (\hat{x} + \hat{y})}$$

3



$\Longleftrightarrow$



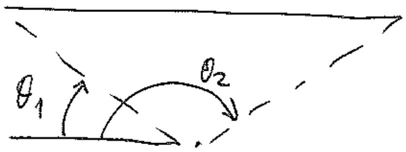
a)

$$\boxed{I_1 = \frac{V}{R} \qquad I_2 = \frac{V}{2R}}$$

Both V and R are given, so that's it!

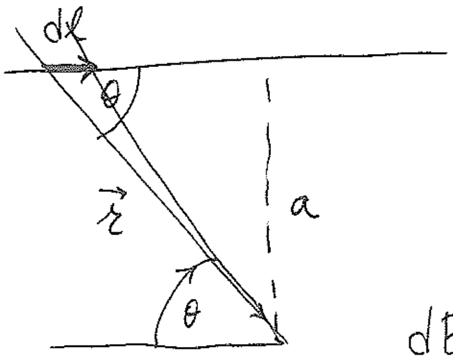
b) The second branch is a voltage divider, so $V_3 = \frac{V}{2}$

c)



For our case $\theta_1 = \frac{\pi}{6}$ $\theta_2 = \frac{5\pi}{6}$

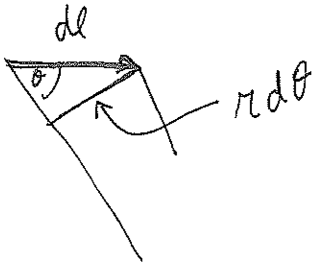
$\left(\theta_2 = \pi - \frac{\pi}{6}, \text{ or } \frac{\pi}{6} + \frac{2\pi}{3}\right)$



Biot-Savart Law:

$$d\vec{B} = \frac{kI}{c}\frac{d\vec{\ell} \times \hat{r}}{r^2}$$

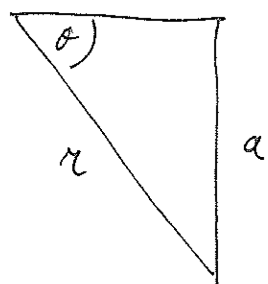
$$dB = \frac{kI}{c}\frac{d\ell \sin\theta}{r^2}$$



$\longrightarrow$ $d\ell \sin\theta = r\,d\theta$,

$r\sin\theta = a$

$\Rightarrow r = \frac{a}{\sin\theta}$



$$\Rightarrow dB = \frac{kI}{c}\frac{\cancel{r}\,d\theta}{r^{\cancel{2}}} = \frac{kI}{c}\frac{d\theta}{\frac{a}{\sin\theta}} = \frac{kI}{ac}\sin\theta\,d\theta$$

For infinite wire $\theta_1 = 0$, $\theta_2 = \pi$ $\int_0^\pi \sin\theta\,d\theta = 2 \Rightarrow B = \frac{2kI}{ac}$

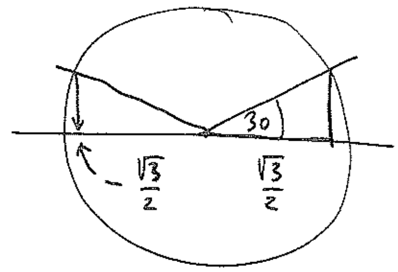
$\boxed{3}$ $\boxed{\text{cont'd}}$

But in our case is $\theta_1 = \frac{\pi}{6}$ and $\theta_2 = \left(\pi - \frac{\pi}{6}\right)$

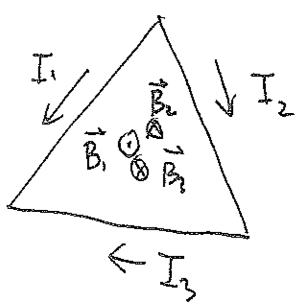
so

$$B = \frac{kI}{ac}\left(\cos\theta_1 - \cos\theta_2\right) = \frac{kI}{ac}\left[\cos\left(\frac{\pi}{6}\right) - \cos\left(\pi - \frac{\pi}{6}\right)\right]$$

$$B = \frac{kI}{ac}\left(\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right)\right) = \frac{kI}{ac}\sqrt{3}$$



The field in the center is the sum of $\vec{B}_1, \vec{B}_2, \vec{B}_3$



$B_1 = \sqrt{3}\,\frac{kI_1}{ac}$ out of paper

$B_2 = B_3 = \sqrt{3}\,\frac{kI_2}{ac}$ into paper.

$$B = B_1 - B_2 - B_3 = \sqrt{3}\,\frac{k}{ac}\left(I_1 - I_2 - \underset{\displaystyle I_3}{\overset{\displaystyle I_2}{\shortparallel}}\right) = \sqrt{3}\,\frac{k}{ac}\left(I_1 - 2I_2\right)$$

But, $I_2 = \frac{1}{2} I_1 \Rightarrow \boxed{B = 0}$!

So we didn't need to do the integral, after all!