Surfaces, Lines, Gauss & Stokes

Recall Gauss's Thm (Divergence Thm):

$$\int (\vec{\nabla}\cdot\vec{A})\, d\tau = \oint \vec{A}\cdot d\vec{a} \qquad (1)$$

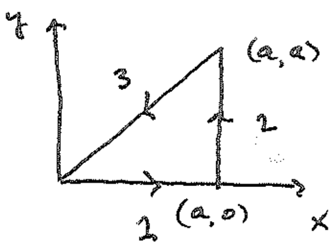
&

Stokes Thm (Curl Thm):

$$\int (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{l} \qquad (2)$$

Here I present examples of the utility of these thms.

I.a) Evaluate $\oint \vec{A}\cdot d\vec{l}$ along the path shown for $\vec{A} = 2xy\,\hat{x} + x^2\hat{y}$



$$\oint \vec{A}\cdot d\vec{l} = \int_1 \vec{A}\cdot d\vec{l} + \int_2 \vec{A}\cdot d\vec{l} + \int_3 \vec{A}\cdot d\vec{l}$$

$$= \int_{0,0}^{a,0} \vec{A}\cdot d\hat{x} + \int_{a,0}^{a,a} \vec{A}\cdot d\hat{y} + \int_{a,a}^{0,0} \vec{A}\cdot(d\hat{x} + d\hat{y})$$

$\int_1 (\ ) = 0$ b/c $y = 0$

$\int_2 (\ ) = \int_0^a x^2 dy \big|_{x=a} = a^3$

$\int_3 (\ ) = \int_{a,a}^{0,0} 2xy\,dx + x^2 dy = \int_{a,a}^{0,0} 2x^2 dx + y^2 dy = -\frac{2}{3}a^3 - \frac{1}{3}a^3 = -a^3$

where the equation of the line for 3 is $x = y$.

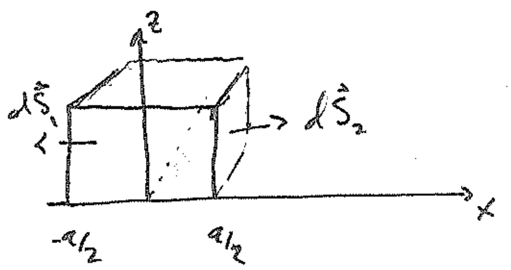
Summing everything $\Rightarrow$ $\boxed{\oint \vec{A}\cdot d\vec{l} = a^3 - a^3 = 0}$

I.b) Use Stokes's Thm to evaluate I.a.

$$\oint \vec{A}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \int \epsilon_{ijk}\partial_j A_k\, d\vec{a} =$$

$$\int \hat{z}\,(\partial_x A_y - \partial_y A_x)\, dx\,dy\,\hat{z} = \int (2x - 2x)\, dx\,dy = 0$$

II. a) Evaluate $\oint x\hat{x}\cdot d\vec{S}$ over surface of cube of side a.



Since $\hat{x}\cdot d\vec{S} = 0$ unless $d\vec{S}$ is in $\hat{x}$ (that is, the yz plane) we only need to evaluate $d\vec{S}_1$ & $d\vec{S}_2$

$$\int x\hat{x}\cdot d\vec{S}_1 = \int (-a/2)\, dz\,dy\ \hat{x}\cdot(-\hat{x}) = a/2\ a^2 = a^3/2$$

$$\int x\hat{x}\cdot d\vec{S}_2 = \int a/2\ dz\,dy\ \hat{x}\cdot(\hat{x}) = a^3/2$$

$$\Rightarrow \oint x\hat{x}\cdot d\vec{S} = \sum_j \int x\hat{x}\cdot d\vec{S}_j = a^3/2 + a^3/2 = a^3$$

II. b) Use Gauss to do above

$$\oint \vec{A}\cdot d\vec{a} = \int(\nabla\cdot\vec{A})\, d\tau = \int (\partial_x x)\, d\tau = a^3$$

For completeness we'll do one for Gradients, too.

<u>Griffiths</u> <u>1.31 c</u> Check Gradient Thm for $T = x^2 + 4xy + 2yz^3$, $\vec{a} = \vec{0}$ $\vec{b} = \vec{1}$, & path $z = x^2$; $y = x$

Gradient Thm: $\int_{\vec{a}\,P}^{\vec{b}} (\nabla T)\cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$

a) $\int_{\vec{a}\,P}^{\vec{b}} (\nabla T)\cdot d\vec{l} = \int_{\vec{0}\,P}^{\vec{1}} (\nabla T)\cdot d\vec{l}$

$$\nabla T = (2x + 4y)\hat{x} + (4x + 2z^3)\hat{y} + 6yz^2\,\hat{z}$$

$$(\nabla T)\cdot d\vec{l} = (2x + 4y)\,dx + (4x + 2z^3)\,dy + 6yz^2\,dz$$

using P: $z = x^2$, $y = x$, $dz = 2x\,dx$ $dy = dx$ $\Rightarrow$

$$\int_P (\ ) = \int_0^1 6x\,dx + (4x + 2x^6)\,dx + 12x^6\,dx$$

$$= \int_0^1 (10x + 14x^6)\,dx = 5x^2 + 2x^7\Big|_0^1 = 7$$

Checking : $T(\vec{b}) - T(\vec{a}) = 1^2 + 4\cdot 1\cdot 1 + 2\cdot 1\cdot 1^3 = 7$

More w/ Gauss & Stokes : IBP.

Griffiths 1.35 :

a) Show $\int_S f(\vec\nabla\times\vec A)\cdot d\vec a = \int_S [\vec A \times \nabla f\,]\cdot d\vec a + \oint_P f\vec A\cdot d\vec l$

$$\int_S f(\vec\nabla\times\vec A)\cdot d\vec a = \int_S f\,\epsilon_{ijk}\,\partial_j A_k\, da_i$$

$$= \int_S \epsilon_{ijk}\,\partial_j(fA_k)\,da_i - \underbrace{\int \epsilon_{ijk} A_k\,\partial_j f\,da_i}_{= \; -\int(\nabla f\times\vec A)\cdot da_i}$$

$\downarrow$ Stokes

$$\oint f\vec A\cdot d\vec l$$

Qed

b) $\int_V \vec B\cdot(\vec\nabla\times\vec A)\,d\tau = \int_V \vec A\cdot(\vec\nabla\times\vec B)\,d\tau + \oint(\vec A\times\vec B)\cdot d\vec a$

$$\int_V \vec B\cdot(\vec\nabla\times\vec A)\,d\tau =$$

$$\int_V B_i\,\epsilon_{ijk}\,\partial_j A_k\,d\tau =$$

$$\int_V \epsilon_{ijk}\,\partial_j(B_i A_k)\,d\tau - \int \epsilon_{ijk}\,A_k\,\partial_j B_i\,d\tau$$

$$\int_V \epsilon_{jki}\,\partial_j(A_k B_i)\,d\tau - \; -\int \epsilon_{jik}(\partial_j B_i)A_k\,d\tau$$

$$= \int_V \nabla\cdot(A\times B)\,d\tau + \int_V(\vec\nabla\times\vec B)\cdot\vec A\,d\tau$$

$\downarrow$ Gauss

$$\oint(\vec A\times\vec B)\cdot d\vec a$$

thus we have

$$\int_V \vec A\cdot(\vec\nabla\times\vec B)\,d\tau + \oint(\vec A\times\vec B)\cdot d\vec l$$