171.304 Quantum Mechanics II. May 8, 2009

Final Exam

There are five problems in the exam. Only the enclosed formula sheets can be used. Each problem is worth 20 points.

Good luck!

1. Consider a helium atom. First, neglect the repulsive potential energy between the electrons and denote the one electron spatial wave functions by ψ_{nlm}.

   (a) Write the two electron states for both the ground state and the excited states involving the one particle spatial states: ψ_{100} and ψ_{21m}. Determine the energies of the states as well.

   (b) Consider now the effect of the repulsion between the electrons ($H_1 = \frac{e^2}{|\mathbf{x}_1 - \mathbf{x}_2|}$) on the excited states studied in (a). Determine which state (singlet or triplet) has the lower energy and why. Give the expression which represents the energy separation between the singlet and triplet states. (The integrals do not have to be calculated.)

2. Consider a tritium atom ($^3\mathbf{H}$) with the electron in the ground state. At $t = 0$ a neutron in the nucleus β^--decays and the atom becomes a $\mathbf{He}^+$ ion. (Since the change was sudden at $t = 0$, the state of the electron did not change; only the Hamiltonian is different for $t \geq 0$.)

   (a) Calculate the probability that the electron ends up in the new ground state.

   (b) Which of the first excited states can the electron transition to and with what probability?

3. A nonreletivistic particle scattered by a spherical-well potential

$$V(r) = \begin{cases} -V_0 & r < R, \\ 0 & r > R. \end{cases}$$

(a) Assuming that the bombarding energy is sufficiently high, calculate the differential cross section in the Born approximation.

(b) Sketch the shape of the angular distribution (indicating the angular units). Discuss how this result can be used to measure R? (It is helpful to know that the function

$$\left(\frac{\sin x - x\cos x}{x^3}\right)^2$$

has its first zero at $x_0 = 1.43\pi$.)

4. A particle of mass m is scattered by a central potential

$$V(r) = -\frac{\hbar^2}{ma^2}\frac{1}{\cosh^2 r/a},$$

where a is a constant.
Given that the equation

$$\frac{d^2y}{dx^2} + \kappa^2 y + \frac{2}{\cosh^2 x} y = 0$$

has the following two solutions:

$$y_\pm = e^{\pm i\kappa x}(\tanh x \mp i\kappa),$$

calculate the s-wave contribution to the total cross section at energy E.

5. A particle of charge e is *confined* to a 3 dimensional box of side a. An electric field $\mathbf{E}$ given by

$$\mathbf{E} = \begin{cases} 0 & t < 0 \\ \mathbf{E}_0 e^{-\alpha t} & t > 0 \end{cases}$$

is applied to the system ($\alpha > 0, H_1 = -e\mathbf{x} \cdot \mathbf{E}$.) $\mathbf{E}_0$ is perpendicular to one of the boundary surfaces of the box.

To lowest order in $|\mathbf{E}_0|$, determine the probability that the charged particle is in the first excited state at $t = \infty$ if it was in the ground state at $t < 0$.

FORMULAE

Laplacian in polar coordinates

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{-\mathbf{L}^2}{\hbar^2 r^2} f,$$

$$\mathbf{L}^2 \longrightarrow -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

Radial equations

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r)\right] R_{E,l}(r) = E R_{E,l}(r)$$

$$u(r) = rR(r),$$

$$-\frac{\hbar^2}{2\mu}\frac{d^2 u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\right] u = Eu$$

$$u(0) = 0$$

Spherical harmonics

$$\int d\Omega Y^*_{l,m}(\theta,\phi) Y_{l',m'}(\theta\phi) = \delta_{l,l'}\delta_{m,m'}$$

$$Y_{0,0} = \sqrt{\frac{1}{4\pi}}, \qquad Y_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}, \qquad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

Hydrogenic atoms

$$E_n = -\frac{\mu c^2 Z^2 \alpha^2}{2n^2}, \qquad R_{1,0} = 2\left(\frac{Z}{a}\right)^{\frac{3}{2}} e^{-Z\frac{r}{a}} \qquad R_{2,0} = 2\left(\frac{Z}{2a}\right)^{\frac{3}{2}}\left(1 - \frac{Zr}{2a}\right) e^{-Z\frac{r}{2a}},$$

$$R_{2,1} = \frac{1}{\sqrt{3}} \left(\frac{Z}{2a}\right)^{\frac{3}{2}} \frac{Zr}{a} e^{-Z\frac{r}{2a}}, \qquad a = \frac{\hbar}{\mu c \alpha}, \qquad \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$$

Spherical Bessel equation

$$x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \left(x^2 - l(l+1)\right) y = 0$$

Solutions are spherical Bessel ($j_l(x)$) and spherical Neumann ($n_l(x)$) functions:

$$j_0(x) = \frac{\sin x}{x}, \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \qquad j_2(x) = \left(\frac{3}{x^2} - \frac{1}{x}\right) \sin x - \frac{3\cos x}{x^2},$$

$$n_0(x) = \frac{-\cos x}{x}, \qquad n_1(x) = \frac{-\sin x}{x} - \frac{\cos x}{x^2}, \qquad n_2(x) = -\left(\frac{3}{x^2} - \frac{1}{x}\right) \cos x - \frac{3\sin x}{x^2},$$

Stationary Non-degenerate Perturbation Theory

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

$$E_n = E_n^{(0)} + < \phi_n^{(0)} | \hat{H}_1 | \phi_n^{(0)} > + \sum_{m \neq n} \frac{| < \phi_n^{(0)} | \hat{H}_1 | \phi_m^{(0)} > |^2}{E_n^{(0)} - E_m^{(0)}} + \ldots$$

Time-dependent Perturbation Theory

$$\hat{H} = \hat{H}_0 + \hat{H}_1(t)$$

Initial state: $|\psi(-\infty) >= |E_k^{(0)} >, \quad |\psi(t) >= \sum_n c_n(t) \exp(-iE_n^{(0)} t/\hbar) |E_n^{(0)} >$

$$c_n(t) = \delta_{n,k} - \frac{i}{\hbar} \int_{-\infty}^{t} dt' \exp i(E_n^{(0)} - E_k^{(0)}) t'/\hbar < E_n^{(0)} | \hat{H}_1(t') | E_k^{(0)} > + \ldots$$

Partial wave expansion

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)\frac{e^{i\delta_l}}{k}\sin\delta_l P_l(\cos\theta); \qquad \frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

$$\sigma = \int d\Omega |f(\theta)|^2 = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l = \sum_{l=0}^{\infty}\sigma_l$$

Born approximation

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int d^3\mathbf{x} V(r)\exp(-i\mathbf{q}\cdot\mathbf{x}), \quad where \quad \mathbf{q} = \mathbf{k}_i - \mathbf{k}_f$$