

# Introduction to Quantum Mechanics 171.303

## Final Exam 12/18/07 9-12

Check the attached formula pages. Start each problem on a fresh page and give detailed reasoning. Please write your name on each page and ask your proctor for clarification if the text is unclear.

### Problem 1 (15 points)

Suppose the wave function

$$\psi(x) = N \exp\left(-\left(\frac{x}{a}\right)^2\right)$$

is known to be a solution to the time independent Schrödinger equation for a particle with mass  $m$  and energy  $E$  in a potential  $V(x)$ . Here  $a$  is a real constant.

- (a) Determine  $N$ . (5 points)
- (b) Determine the potential  $V(x)$  in terms of  $E$ ,  $a$ , and  $m$  and discuss the connection of your result to a well known problem in quantum mechanics. (10 points)

### Problem 2 (20 points)

The spin state of an electron can be described as  $|+\mathbf{z}\rangle$  at time  $t=0$ . A magnetic field of magnitude  $B$  is now applied along a direction  $\hat{\mathbf{n}}$  which forms an angle  $\varphi$  with the  $\hat{\mathbf{z}}$ -axis.

- (a) Write an expression for the initial state of the electron spin in the  $|\pm\mathbf{n}\rangle$  basis. (5 points)
- (b) Determine the time dependence of this spin state following application of the field. (5 points)
- (c) Determine the time dependence of  $\langle S_z \rangle$  and  $\langle S_n \rangle$ . (10 points)

### Problem 3 (20 points)

Consider a system of two spin angular momenta  $\hat{\mathbf{S}}_1$  and  $\hat{\mathbf{S}}_2$ . Define the total angular momentum operator  $\hat{\mathbf{S}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$ .

- (a) Show that it is *not* possible to find simultaneous eigenstates for  $\hat{S}^2$  and  $\hat{S}_{1z}$ . (3 points)
- (b) Show that it *is* possible to find simultaneous eigenstates for  $\hat{S}^2$  and  $\hat{S}_z$ . (5 points)

Denote eigenstates of  $\hat{S}_{1z}$  and  $\hat{S}_{2z}$  by  $|m_1 m_2\rangle$  and eigenstates of  $\hat{S}^2$  and  $\hat{S}_z$  by  $|sm\rangle$  and assume the quantum numbers for the angular momenta involved are  $s_1 = 3/2$  and  $s_2 = 1$ .

- (c) What values of  $s$  are possible? (2 points)
- (c) Derive expressions for all states  $|sm\rangle = |\frac{5}{2}m\rangle$  where  $m = -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$  in terms of the states  $|m_1 m_2\rangle$ . Use raising and/or lowering operators and check your results against the attached table of Clebsch-Gordan coefficients. (10 points)

#### Problem 4 (15 points)

A particle with mass  $m$  is subject to the following potential:

$$V(x) = -\alpha\delta(x+a) + \beta\delta(x) - \alpha\delta(x-a)$$

Here  $\alpha$ ,  $\beta$ , and  $a$  are suitably dimensioned positive constants and  $\delta(x)$  is the Dirac delta function.

- (a) Write expressions for and sketch bound state wave functions for this problem with due consideration for symmetry. (5 points)
- (b) Derive an equation from which the bound state energy for the *odd* wave function can be derived and use a graphical technique to discuss the solution in appropriate limits. (10 points)

#### Problem 5 (15 points)

A particle with mass  $m$  moves in a harmonic potential  $V(x) = \frac{1}{2}m\omega^2 x^2$ . At  $t = 0$  the wave function takes the form:  $|\psi(t=0)\rangle = A(|n\rangle + |n+1\rangle)$ , where  $|n\rangle$  is an energy eigenstates in the usual notation.

- (c) Determine  $A$ . (3 points)
- (d) Determine  $|\psi(t)\rangle$ . (5 points)
- (a) Determine  $\langle x \rangle$  and  $\langle p \rangle$  versus time. (7 points)

#### Problem 6 (15 points)

Use the WKB approximation to obtain an estimate of the bound state energies for a particle with mass  $m$  in the potential  $V(x) = \alpha|x|$  where  $\alpha > 0$  is a real constant.

## Formulae

Simple Harmonic Oscillator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} + \frac{i}{m\omega} \hat{p} \right)$$

$$\hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}} (\hat{a} - \hat{a}^\dagger)$$

$$[\hat{a}, \hat{a}^\dagger] = 1$$

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$$

Raising and lowering operators

$$\hat{S}_\pm |sm\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)} |s, m\pm 1\rangle$$

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Representation of rotation operator for spin-1/2 states

$$R(\phi \hat{\mathbf{n}}) = \mathbf{1} \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} (\vec{\sigma} \cdot \hat{\mathbf{n}})$$

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

WKB bound states

$\int_{x_1}^{x_2} p(x) dx = n\pi\hbar$	Two hard boundaries
$\int_{x_1}^{x_2} p(x) dx = (n - \frac{1}{4})\pi\hbar$	One hard boundary
$\int_{x_1}^{x_2} p(x) dx = (n - \frac{1}{2})\pi\hbar$	No hard boundaries

A potentially useful integral:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

A table of Clebsch-Gordan coefficients is provided