Introduction to Quantum Mechanics 171.303
Final Exam 12/20/05 9-12

Check the attached formula pages. Start each problem on a fresh page and give detailed reasoning. Please write your name on each page and ask your proctor for clarification if the text is unclear.

Problem 1 (30 points)

A charged harmonic oscillator with in a uniform electric field $\mathcal{E}$ has the following Hamiltonian operator (position basis):

$$\hat{H}_{\mathcal{E}} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 - \mathcal{E}qx$$

(a) By re-writing the potential energy term in a quadratic form show that the eigenvalues and eigenstates can be written as follows

$$E_n = \hbar\omega\left(\tfrac{1}{2} + n\right) - \frac{q^2\mathcal{E}^2}{2m\omega^2}$$

$$\left|n\right\rangle_{\mathcal{E}} = \hat{T}(d)\left|n\right\rangle_{\mathcal{E}=0}$$

Here $\hat{T}(d) = \exp\left(-i\hat{p}_x d/\hbar\right)$ is the translation operator and $d = \frac{q\mathcal{E}}{m\omega^2}$ (8 points)

Now assume that the particle is in the ground state $\left|0\right\rangle_{\mathcal{E}}$ when the electric field is abruptly removed. Denote by P_n the probability that a measurement of the energy of the system following that treatment will yield the value $\hbar\omega\left(\tfrac{1}{2} + n\right)$.

(b) In the following you will be guided towards and analytical expression for P_n. To start off, derive an expression for P_n that involves $\hat{T}(d)$. (4 points)

(c) Show that $\exp(\lambda\hat{a})\left|0\right\rangle = \left|0\right\rangle$. Here $\hat{a}$ is the lowering operator for the harmonic oscillator. (4 points)

(d) Show that $\left\langle n\right|\exp\left(\lambda\hat{a}^+\right)\left|0\right\rangle = \frac{\lambda^n}{\sqrt{n!}}$. Here $\hat{a}^+$ is the raising operator for the harmonic oscillator. (6 points)

(e) It can be shown that $\exp(\hat{A})\exp(\hat{B}) = \exp(\hat{A}+\hat{B})\exp\left(\tfrac{1}{2}[\hat{A},\hat{B}]\right)$ when both operators $\hat{A}$ and $\hat{B}$ commute with $[\hat{A},\hat{B}]$. Use this result and the results from (b)-(d) to show that

$$P_n = \exp\left(-\lambda^2\right)\frac{1}{n!}\left(\lambda^2\right)^n \quad \text{where } \lambda = d\sqrt{\frac{m\omega}{2\hbar}}$$

(8 points)

Problem 2 (15 points)

The Hamiltonian for a certain three level system and operators $\hat{A}$ and $\hat{B}$ are represented by the following matrixes:

$$\hat{H} \rightarrow \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \hat{A} \rightarrow \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \hat{B} \rightarrow \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here λ and μ are real positive numbers.

(a) Show that $\hat{A}$ and $\hat{B}$ are physical observables. (3 points)

(b) Are there simultaneous eigenstates for $\hat{H}$ and $\hat{A}$? Give detailed reasoning. (4 points)

(c) Do states exist for which $\langle A \rangle$ is time independent? Give detailed reasoning. (3 points)

(d) Establish a lower bound on the products of uncertainties $\Delta A \Delta B$ for measurements of A and B in the following quantum state:

$$|\psi\rangle \rightarrow \frac{1}{\sqrt{3}} \begin{pmatrix} i \\ 1 \\ -i \end{pmatrix}$$

(5 points)

Problem 3 (15 points)

Consider two particles with spin quantum number s_1=1 and s_2=2 respectively. Their interaction can be described by the Hamilton operator $\hat{H} = 2\lambda \hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2$ where $\lambda > 0$.

(a) By rewriting $\hat{H}$ in terms of $\hat{S}^2 = (\hat{\mathbf{s}}_1 + \hat{\mathbf{s}}_2)^2$ compute the eigenvalues of $\hat{H}$ and the number of degenerate states for each eigenvalue. (7 points)

The system is in its lowest energy state when a measurement of the projection of the total angular momentum along the z-direction is performed. The measurement yields $S_z = \hbar$.

(b) Determine the possible results for a subsequent measurement of s_{2z} and the respective probabilities. (8 points)

Problem 4 (18 points)

Consider one dimensional motion of a particle with mass m in the following potential: $V(x) = -\alpha\delta(x) + \beta\theta(x)$, where $\alpha > 0$ and $\beta > 0$. Also $\theta(x)$ denotes the step function which is zero for $x < 0$ and one otherwise.

(a) Calculate and sketch the transmission and reflection coefficients as a function of energy, E, for a particle entering from $x = -\infty$ with $E > 0$. (12 points)

(b) Determine whether a bound state exists and if so compute the energy of that state. (6 points)

Problem 5 (12 points)

Consider a spin-1 particle with mass m, charge q, and g-factor, g. Throughout this problem, the particle is initially prepared in a state $|\psi\rangle$ about which we know that a measurement of the z-projection of angular momentum has equal probability of yielding the values $\hbar$, 0, and $-\hbar$. (Note that this does not fully specify $|\psi\rangle$).

(a) Determine the largest and the smallest possible values of $|\langle S_x \rangle|$ for such a state. (4 points)

In the following two questions a magnetic field B is applied along the $\hat{z}$-axis.

(b) Describe why this does or does not affect the probabilities for the outcome of a measurements of S_z. (4 points)

(c) Determine the time dependence of $\langle S_x \rangle$ when the initial state $|\psi\rangle$ maximizes $\langle S_x \rangle$ within the constraints described above. (4 points)

Problem 6 (10 points)

A particle with mass m is trapped in an infinite square well potential:

$$V(x) = \begin{cases} 0 \text{ for } 0 < x < L \\ \infty \quad \text{otherwise} \end{cases}$$

At time t=0 the wave function of the particle takes the following form:

$$\psi(x, t=0) = \begin{cases} Nx & \text{for } 0 < x < L/2 \\ N(L-x) & \text{for } L/2 < x < L \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the normalization constant N. (4 points)

(b) Determine the probability that a measurement of the energy of the particle yields the ground state energy of the well. (6 points)

Formulae

Simple Harmonic Oscillator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$

$$\hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a} - \hat{a}^{+}\right)$$

$$[\hat{a}, \hat{a}^{+}] = 1$$

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^{+}\right)^{n}|0\rangle$$

Raising and lowering operators

$$\hat{S}_{\pm}|sm\rangle = \hbar\sqrt{s(s+1) - m(m \pm 1)}\,|s, m \pm 1\rangle$$

Spin-1 Representations

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \hat{S}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

A table of Clebsch-Gordan coefficients is provided