Introduction to Quantum Mechanics 171.303
Midterm Exam 11/1/07 1-2

Check the attached formula pages. Start each problem on a fresh page and give detailed reasoning. Please ask your proctor for clarification if the text is unclear.

Problem 1 (25 points)

A beam of spin-1/2 particles each prepared in the specific quantum state $|\psi\rangle$ is incident on a Stern-Gerlach apparatus with field gradient along $\hat{\mathbf{z}}$ (SG**z** apparatus). Equal numbers of particles are detected from the two channels.

(a) Write a general expression for $|\psi\rangle$ in the $|\pm\mathbf{z}\rangle$ basis with suitably chosen parameters expressing aspects of the wave function that cannot be determined on the basis of this experiment. (10 points)

To learn more about $|\psi\rangle$ the particles are now subject to a magnetic field **B** along $\hat{\mathbf{x}}$ for a fixed time T before entering the SG**z** apparatus. For suitable field strengths it is found that no particles emerge from the $+\hbar/2$ channel of the SG**z** apparatus.

(b) Use this information to further specify $|\psi\rangle$. (15 points)

Problem 2 (25 points)

Operators $\hat{A}$ and $\hat{B}$ have the following representations in the basis of states $|1\rangle$ and $|2\rangle$:

$$\hat{A} \to a\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \hat{B} \to b\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}.$$

Here *a* and *b* are suitably dimensioned real constants.

(a) For each operator determine whether it can be associated with a physical observable. (5 points)

(b) Do quantum states exist for which the outcome of a measurement of both physical observables can be predicted with certainty? (10 points)

(c) Determine the mean value and the variance in a measurement of *A* on a particle in the basis state $|1\rangle$. (10 points)

Problem 3 (30 points)

An isotropic composite object is composed of a spin-2 particle and a spin-3/2 particle arranged in a state of lowest possible angular momentum.

(a) Write a Hamiltonian involving the angular momentum operators for the two particles for which the state described above is the ground state. (10 points)

The objects now breaks apart into its constituent particles and a measurement of the projection of angular momentum along $\hat{\mathbf{z}}$ is carried out for the spin-2 particle.

(b) Determine what values can occur and their respective probabilities. (10 points)

(c) If $\hbar$ is the result of the measurement, what are the possible values and probabilities for a subsequent measurement of the projection of angular momentum along $\hat{\mathbf{z}}$ for the spin-3/2 particle. Hint: use the table of Clebsch-Gordan coefficients. (10 points)

Problem 4 (20 points)

Consider a three state quantum system with the following Hamiltonian:

$$\hat{\mathcal{H}} \rightarrow \hbar\omega \begin{pmatrix} 7 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 5 \end{pmatrix}$$

(a) Determine the eigenvalues and eigenstates of $\hat{\mathcal{H}}$ and describe their physical significance. (10 points)

Suppose the system is initially prepared in the state

$$|\psi(t=0)\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We are using the same basis as used to represent $\hat{\mathcal{H}}$ above.

(b) Determine the state of the system at a later time, $|\psi(t)\rangle$. (10 points)

Formulae

Raising and lowering operators

$$\hat{S}_{\pm}|sm\rangle = \hbar\sqrt{s(s+1)-m(m\pm 1)}\,|s,m\pm 1\rangle$$

Spin-1/2 eigenstates

$$|+\mathbf{x}\rangle = \frac{1}{\sqrt{2}}\left(|+\mathbf{z}\rangle + |-\mathbf{z}\rangle\right)$$

$$|+\mathbf{y}\rangle = \frac{1}{\sqrt{2}}\left(|+\mathbf{z}\rangle + i|-\mathbf{z}\rangle\right)$$

Spin-1/2 representations

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Representation of rotation operator for spin-1/2 states

$$R(\phi\,\hat{\mathbf{n}}) = \mathbf{1}\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}(\vec{\sigma}\cdot\hat{\mathbf{n}})$$

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

Spin-1 representations

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \hat{S}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

A table of Clebsch-Gordan coefficients is attached.

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, e.g., for −8/15 read $-\sqrt{8/15}$.

Notation:

	J M	J M	...
m_1 m_2	Coefficients		
m_1 m_2			
. .			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1 m_2	$J=1$, $M=+1$
+1/2 +1/2	1

m_1 m_2	$J=1$, $M=0$	$J=0$, $M=0$
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	$J=1$, $M=-1$
−1/2 −1/2	1

1×1/2

m_1 m_2	$J=3/2$, $M=+3/2$
+1 +1/2	1

m_1 m_2	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	$J=3/2$, $M=-3/2$
−1 −1/2	1

2×1

m_1 m_2	$J=3$, $M=+3$
+2 +1	1

m_1 m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	$J=3$, $M=-3$
−2 −1	1

1×1

m_1 m_2	$J=2$, $M=+2$
+1 +1	1

m_1 m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	$J=2$, $M=-2$
−1 −1	1

2×1/2

m_1 m_2	$J=5/2$, $M=+5/2$
+2 +1/2	1

m_1 m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	$J=5/2$, $M=-5/2$
−2 −1/2	1

3/2×1/2

m_1 m_2	$J=2$, $M=+2$
+3/2 +1/2	1

m_1 m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	$J=2$, $M=0$	$J=1$, $M=0$
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	$J=2$, $M=-2$
−3/2 −1/2	1

3/2×1

m_1 m_2	$J=5/2$, $M=+5/2$
+3/2 +1	1

m_1 m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	$J=5/2$, $M=-5/2$
−3/2 −1	1

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

2×3/2

m_1 m_2	$J=7/2$, $M=+7/2$
+2 +3/2	1

m_1 m_2	$J=7/2$, $M=+5/2$	$J=5/2$, $M=+5/2$
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	$J=7/2$, $M=+3/2$	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	$J=7/2$, $M=+1/2$	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	$J=7/2$, $M=-1/2$	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	$J=7/2$, $M=-3/2$	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	$J=7/2$, $M=-5/2$	$J=5/2$, $M=-5/2$
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	$J=7/2$, $M=-7/2$
−2 −3/2	1

3/2×3/2

m_1 m_2	$J=3$, $M=+3$
+3/2 +3/2	1

m_1 m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	$J=3$, $M=-3$
−3/2 −3/2	1

2×2

m_1 m_2	$J=4$, $M=+4$
+2 +2	1

m_1 m_2	$J=4$, $M=+3$	$J=3$, $M=+3$
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	$J=4$, $M=+2$	$J=3$, $M=+2$	$J=2$, $M=+2$
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	$J=4$, $M=+1$	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	$J=4$, $M=0$	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	$J=4$, $M=-1$	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	$J=4$, $M=-2$	$J=3$, $M=-2$	$J=2$, $M=-2$
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	$J=4$, $M=-3$	$J=3$, $M=-3$
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	$J=4$, $M=-4$
−2 −2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta - 1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta + 1}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1) \qquad d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.