Introduction to Quantum Mechanics 171.303
Midterm Exam 11/3/05 1-2

Check the attached formula pages. Start each problem on a fresh page and please give detailed reasoning. Please ask your proctor for clarification if the text is unclear.

Problem 1 (30 points)

Consider a spin-1/2 particle with charge q and mass m, in a magnetic field of strength B oriented in the $\hat{\mathbf{z}}$ direction. At t=0, the particle is in the state

$$|\psi(0)\rangle = A\left(|-\mathbf{x}\rangle - |+\mathbf{z}\rangle\right)$$

(a) Compute the normalization constant A. (10 points)

(b) Compute $\langle S_z \rangle$ at t=0. (10 points)

(c) Compute $\langle S_y \rangle$ as a function of t. (10 points)

Problem 2 (20 points)

Consider a spin-3/2 particle which we shall describe in the basis of eigenstates for $\hat{S}_z$.

(a) Write the matrix representation for $\hat{S}_z$ and $\hat{S}_+$ (10 points)

(b) Write the matrix representations for $\hat{S}_x$ (10 points)

Problem 3 (20 points)

Consider two particles with spin quantum number S_1=3/2 and S_2=1 respectively. Their interaction can be described by a spin Hamiltonian $\hat{\mathcal{H}} = 2\hbar^{-2} J\, \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$.

(a) Show that there exist simultaneous eigenstates of $\hat{\mathcal{H}}$ and $\hat{S}^2_{tot} = \left(\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2\right)^2$ (10 points)

(b) Compute the eigenvalues of $\hat{\mathcal{H}}$ and the number of degenerate states for each energy level. (10 points)

Problem 4 (30 points)

Consider a spin-1/2 particle that we describe in two different bases. The basis B_1 consists of eigenstates for $\hat{S}_z$ and basis B_2 consists of eigenstates for $\hat{S}_n$. The spherical coordinates for the unit vector $\hat{\mathbf{n}}$ are $(r\theta\phi) = (1\theta\phi)$ so that a rotation of ϕ about $\hat{\mathbf{z}}$ followed by a rotation of θ about $\hat{\mathbf{y}}$ brings $\hat{\mathbf{z}}$ into coincidence with $\hat{\mathbf{n}}$.

(a) Write the representation of $\hat{S}^2$ and $\hat{S}_n$ in basis B_2. (8 poins)

(b) Compute the transformation from the representation of a general state $|\psi\rangle$ in basis B_1 to the representation of $|\psi\rangle$ in basis B_2. (12 points)

(c) Compute the representation of $\hat{S}_z$ in basis B_2 and check your result by selecting appropriate values of θ and ϕ. (10 points)

Formulae

Raising and lowering operators

$$\hat{S}_{\pm}|sm\rangle = \hbar\sqrt{s(s+1)-m(m\pm 1)}\,|s,m\pm 1\rangle$$

Spin-1/2 eigenstates

$$|+\mathbf{x}\rangle = \frac{1}{\sqrt{2}}\left(|+\mathbf{z}\rangle + |-\mathbf{z}\rangle\right)$$

$$|+\mathbf{y}\rangle = \frac{1}{\sqrt{2}}\left(|+\mathbf{z}\rangle + i|-\mathbf{z}\rangle\right)$$

Spin-1/2 Representations

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli Matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Representation of rotation operator for spin-1/2 states

$$R(\phi\hat{\mathbf{n}}) = \mathbf{1}\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}(\vec{\sigma}\cdot\hat{\mathbf{n}})$$

Spin-1 Representations

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \hat{S}_z = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

A table of Clebsch-Gordan coefficients is attached.