Introduction to Quantum Mechanics 171.304
Midterm Exam 3/30/06 1-2

Check the attached formula pages. Start each problem on a fresh page and give detailed reasoning. Please ask your proctor for clarification if the text is unclear.

Problem 1 (30 points)

Consider the *n=4* level of the hydrogen atom.

(a) Neglecting any perturbations how many degenerate states are there? (10 points)
(b) In the presence of a strong magnetic field and neglecting fine structure, how many distinct energy levels exist and what is the degeneracy of each of these levels. (10 points)
(c) Now consider the fine structure perturbations in zero external magnetic field. How many distinct energy levels exist and what is the degeneracy of each of these levels? (10 points)

Problem 2 (20 points)

Consider a three state system with unperturbed Hamiltonian

$$\hat{H}_0 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Add a small perturbation

$$\hat{H}_1 = \begin{pmatrix} -\alpha & -i\alpha & 0 \\ i\alpha & \alpha & i\alpha \\ 0 & -i\alpha & \alpha \end{pmatrix},$$

where $\alpha << 1$. Compute the leading corrections to the three energy eigenvalues.

Problem 3 (25 points)

Consider the following variational wave function for the hydrogen atom:

$$\psi_a(r\theta\phi) = \begin{cases} C(a-r) & \text{for } r \le a \\ 0 & \text{otherwise} \end{cases}$$

Determine the value of *a* that minimizes the total energy of the state and compare your result to the Bohr radius: $a_0 = \frac{\hbar^2}{me^2}$.(Hint: Use the Laplacian in spherical coordinates which is on the next page)

Problem 4 (25 points)

Consider two identical charged spin-1 particles in a central Coulomb potential. You can neglect spin orbit coupling and the Coulomb repulsion between the two spin-1 particles can be considered a weak perturbation.

(a) Why is the total spin a conserved quantity for this system? (5 points)
(b) Give detailed reasoning for why the ground state has a total spin quantum number of 0 or 2. (Hint: Consult the table of Clebsch-Gordan coefficients) (10 points)
(c) Argue why the first excited two particle state must have a total spin quantum number of 1. (10 points)

Formulae

Laplace operator in spherical coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

A table of Clebsch-Gordan coefficients is provided

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

m_1	m_2	J / M
m_1	m_2	Coefficients

$1/2 \times 1/2$

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	-1/2	1/2	1/2
-1/2	+1/2	1/2	-1/2

m_1	m_2	J=1, M=-1
-1/2	-1/2	1

$1 \times 1/2$

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	-1/2	1/3	2/3
0	+1/2	2/3	-1/3

m_1	m_2	J=3/2, M=-1/2	J=1/2, M=-1/2
0	-1/2	2/3	1/3
-1	+1/2	1/3	-2/3

m_1	m_2	J=3/2, M=-3/2
-1	-1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	-1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	-1	1/15	1/3	3/5
+1	0	8/15	1/6	-3/10
0	+1	2/5	-1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	-1	1/5	1/2	3/10
0	0	3/5	0	-2/5
-1	+1	1/5	-1/2	3/10

m_1	m_2	J=3, M=-1	J=2, M=-1	J=1, M=-1
0	-1	2/5	1/2	1/10
-1	0	8/15	-1/6	-3/10
-2	+1	1/15	-1/3	3/5

m_1	m_2	J=3, M=-2	J=2, M=-2
-1	-1	2/3	1/3
-2	0	1/3	-2/3

m_1	m_2	J=3, M=-3
-2	-1	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	-1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	-1	1/6	1/2	1/3
0	0	2/3	0	-1/3
-1	+1	1/6	-1/2	1/3

m_1	m_2	J=2, M=-1	J=1, M=-1
0	-1	1/2	1/2
-1	0	1/2	-1/2

m_1	m_2	J=2, M=-2
-1	-1	1

Spherical harmonics

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$2 \times 1/2$

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	-1/2	1/5	4/5
+1	+1/2	4/5	-1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	-1/2	2/5	3/5
0	+1/2	3/5	-2/5

m_1	m_2	J=5/2, M=-1/2	J=3/2, M=-1/2
0	-1/2	3/5	2/5
-1	+1/2	2/5	-3/5

m_1	m_2	J=5/2, M=-3/2	J=3/2, M=-3/2
-1	-1/2	4/5	1/5
-2	+1/2	1/5	-4/5

m_1	m_2	J=5/2, M=-5/2
-2	-1/2	1

$3/2 \times 1/2$

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	-1/2	1/4	3/4
+1/2	+1/2	3/4	-1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	-1/2	1/2	1/2
-1/2	+1/2	1/2	-1/2

m_1	m_2	J=2, M=-1	J=1, M=-1
-1/2	-1/2	3/4	1/4
-3/2	+1/2	1/4	-3/4

m_1	m_2	J=2, M=-2
-3/2	-1/2	1

$3/2 \times 1$

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	-2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	-1	1/10	2/5	1/2
+1/2	0	3/5	1/15	-1/3
-1/2	+1	3/10	-8/15	1/6

m_1	m_2	J=5/2, M=-1/2	J=3/2, M=-1/2	J=1/2, M=-1/2
+1/2	-1	3/10	8/15	1/6
-1/2	0	3/5	-1/15	-1/3
-3/2	+1	1/10	-2/5	1/2

m_1	m_2	J=5/2, M=-3/2	J=3/2, M=-3/2
-1/2	-1	3/5	2/5
-3/2	0	2/5	-3/5

m_1	m_2	J=5/2, M=-5/2
-3/2	-1	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

$3/2 \times 3/2$

m_1	m_2	J=3, M=+3
+3/2	+3/2	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	-1/2

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	-1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	-2/5
-1/2	+3/2	1/5	-1/2	3/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	-3/2	1/20	1/4	9/20	1/4
+1/2	-1/2	9/20	1/4	-1/20	-1/4
-1/2	+1/2	9/20	-1/4	-1/20	1/4
-3/2	+3/2	1/20	-1/4	9/20	-1/4

m_1	m_2	J=3, M=-1	J=2, M=-1	J=1, M=-1
+1/2	-3/2	1/5	1/2	3/10
-1/2	-1/2	3/5	0	-2/5
-3/2	+1/2	1/5	-1/2	3/10

m_1	m_2	J=3, M=-2	J=2, M=-2
-1/2	-3/2	1/2	1/2
-3/2	-1/2	1/2	-1/2

m_1	m_2	J=3, M=-3
-3/2	-3/2	1

$2 \times 3/2$

m_1	m_2	J=7/2, M=+7/2
+2	+3/2	1

m_1	m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	-3/7

m_1	m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	-1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	-2/5
0	+3/2	2/7	-18/35	1/5

m_1	m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	-3/2	1/35	6/35	2/5	2/5
+1	-1/2	12/35	5/14	0	-3/10
0	+1/2	18/35	-3/35	-1/5	1/5
-1	+3/2	4/35	-27/70	2/5	-1/10

m_1	m_2	J=7/2, M=-1/2	J=5/2, M=-1/2	J=3/2, M=-1/2	J=1/2, M=-1/2
+1	-3/2	4/35	27/70	2/5	1/10
0	-1/2	18/35	3/35	-1/5	-1/5
-1	+1/2	12/35	-5/14	0	3/10
-2	+3/2	1/35	-6/35	2/5	-2/5

m_1	m_2	J=7/2, M=-3/2	J=5/2, M=-3/2	J=3/2, M=-3/2
0	-3/2	2/7	18/35	1/5
-1	-1/2	4/7	-1/35	-2/5
-2	+1/2	1/7	-16/35	2/5

m_1	m_2	J=7/2, M=-5/2	J=5/2, M=-5/2
-1	-3/2	4/7	3/7
-2	-1/2	3/7	-4/7

m_1	m_2	J=7/2, M=-7/2
-2	-3/2	1

2×2

m_1	m_2	J=4, M=+4
+2	+2	1

m_1	m_2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	-1/2

m_1	m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	-3/7
0	+2	3/14	-1/2	2/7

m_1	m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	-1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	-1/14	-3/10
0	+1	3/7	-1/5	-1/14	3/10
-1	+2	1/14	-3/10	3/7	-1/5

m_1	m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	-2	1/70	1/10	2/7	2/5	1/5
+1	-1	8/35	2/5	1/14	-1/10	-1/5
0	0	18/35	0	-2/7	0	1/5
-1	+1	8/35	-2/5	1/14	1/10	-1/5
-2	+2	1/70	-1/10	2/7	-2/5	1/5

m_1	m_2	J=4, M=-1	J=3, M=-1	J=2, M=-1	J=1, M=-1
+1	-2	1/14	3/10	3/7	1/5
0	-1	3/7	1/5	-1/14	-3/10
-1	0	3/7	-1/5	-1/14	3/10
-2	+1	1/14	-3/10	3/7	-1/5

m_1	m_2	J=4, M=-2	J=3, M=-2	J=2, M=-2
0	-2	3/14	1/2	2/7
-1	-1	4/7	0	-3/7
-2	0	3/14	-1/2	2/7

m_1	m_2	J=4, M=-3	J=3, M=-3
-1	-2	1/2	1/2
-2	-1	1/2	-1/2

m_1	m_2	J=4, M=-4
-2	-2	1

d functions

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.