Quantum Mechanics 171.303

Summary of Fall Semester 2007

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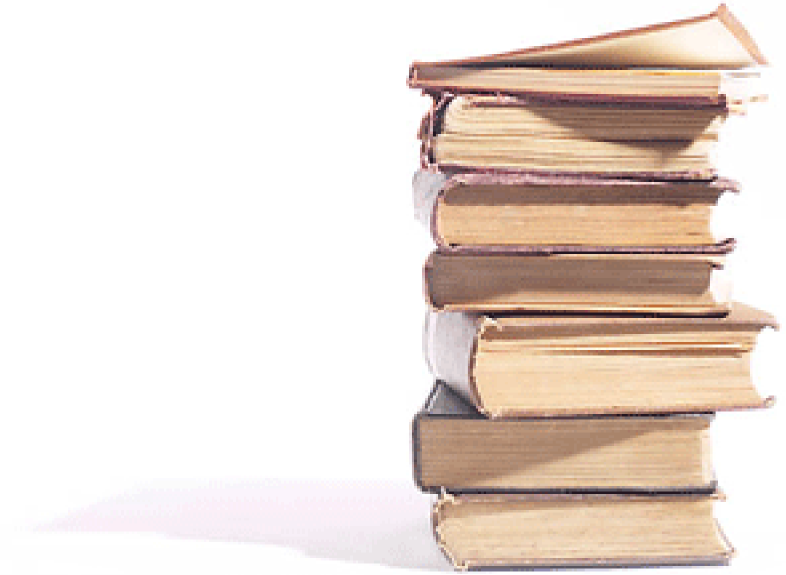
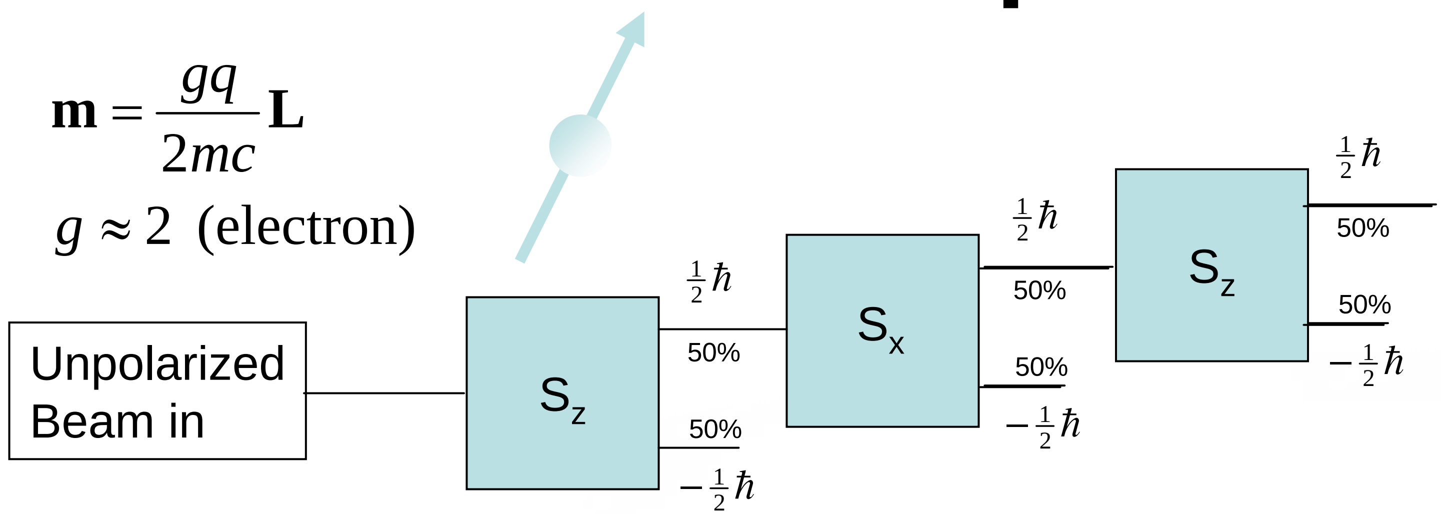


The Stern Gerlach Experiment

$$\mathbf{m} = \frac{gq}{2mc}\mathbf{L}$$

$$g \approx 2 \ \ \text{(electron)}$$



$$|\psi\rangle = c_+|+z\rangle + c_-|-z\rangle$$

Resolve quantum state in components

$$1 = \langle\psi|\psi\rangle = |c_+|^2 + |c_-|^2$$

Normalization for probabilistic interpretation

$|c_+|^2 =$ Probability that measurement of S_z yields $\frac{1}{2}\hbar$

$$1 = |+z\rangle\langle+z| + |-z\rangle\langle-z|$$

Completeness

$$\langle+z|-z\rangle = \langle-z|+z\rangle = 0$$

Orthogonality

Generalize to multidimensional “Hilbert space”

Representations and Operators

$$\left|\pm x\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|+z\right\rangle \pm \left|-z\right\rangle\right) \rightarrow \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm 1\end{pmatrix}$$

$$\left|\pm y\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|+z\right\rangle \pm i\left|-z\right\rangle\right) \rightarrow \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix}$$

"Representation" of Ket in chosen basis

$$\hat{A}\left|\psi\right\rangle = \left|\phi\right\rangle \rightarrow \sum_{ij}\left\langle i\right|\hat{A}\left|j\right\rangle\left\langle j|\psi\right\rangle = \sum_{i}\left\langle i|\phi\right\rangle \qquad \hat{A} \rightarrow \mathbf{A}$$

$$\left\langle\psi\right|\hat{A}^{+} = \left\langle\phi\right| \rightarrow \sum_{ij}\left\langle\phi|j\right\rangle\left\langle j\right|\hat{A}\left|i\right\rangle^{*} = \sum_{i}\left\langle\phi|i\right\rangle \qquad \hat{A}^{+} \rightarrow \left(\mathbf{A}^{+}\right)^{*}$$

$\hat{P}_{\pm} = \left|\pm z\right\rangle\left\langle\pm z\right|$ Projects component : $\hat{P}_{+}\left(c_{+}\left|+z\right\rangle + c_{-}\left|-z\right\rangle\right) = c_{+}\left|+z\right\rangle$

$$\hat{P}_{+} \rightarrow \begin{pmatrix}\left\langle+z\right|\hat{P}_{+}\left|+z\right\rangle & \left\langle+z\right|\hat{P}_{+}\left|-z\right\rangle\\ \left\langle-z\right|\hat{P}_{+}\left|+z\right\rangle & \left\langle-z\right|\hat{P}_{+}\left|-z\right\rangle\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

Change of basis

For ket (shown here for two state basis)

$$\begin{pmatrix} \langle +u|\psi\rangle \\ \langle -u|\psi\rangle \end{pmatrix} = \underbrace{\begin{pmatrix} \langle +u|+z\rangle & \langle +u|-z\rangle \\ \langle -u|+z\rangle & \langle -u|-z\rangle \end{pmatrix}}_{\text{Columns are coordinates of old basis states in new basis}} \begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix} \equiv \mathrm{S}^{+} \begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$$

Columns are coordinates of
old basis states in new basis

For Operator

$$\mathrm{A}(\text{new basis}) = \mathrm{S}^{+}\,\mathrm{A}(\text{old basis})\,\mathrm{S}$$

New and old basis states are related by unitary transformation

$|\pm u\rangle = \hat{R}|\pm z\rangle$ (where $\hat{R}^{+}\hat{R} = 1$ to preserve normalized states)

Then **S** is representation of $\hat{R}$ in new and old basis

Rotation, Angular Momentum & observables

Unitary Rotation operator: $\hat{R}(\phi\hat{\mathbf{n}}) = \exp(-i\hat{J}_n\phi/\hbar)$

$\hat{J}_n$ Is Hermitian ($\hat{J}_n^{\,+} = \hat{J}_n$) with dimension of angular momentum

We determine the action of $\hat{R}(\phi\hat{\mathbf{n}})$ through Taylor expansion & find

$\hat{J}_n|\pm n\rangle = \pm\tfrac{1}{2}\hbar|\pm n\rangle$ ⟹ Eigen states $|\pm n\rangle$
Eigen values $\pm\tfrac{1}{2}\hbar$

Infer that $\hat{J}_n$ is angular momentum projection operator

$$\hat{S}_x \to \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y \to \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z \to \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\langle J_x\rangle = \langle\psi|\hat{J}_x|\psi\rangle = \left(\langle\psi|+z\rangle, \langle\psi|-z\rangle\right)\tfrac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$$

Angular Momentum Algebra

Order of rotations matter which in operator language implies:

$$\left[\hat{R}(\delta\phi\,\hat{x}),\hat{R}(\delta\phi\,\hat{y})\right]=\hat{R}(\delta\phi\,\hat{z})-1$$

This implies that angular momentum operators don't commute:

$$\left[\hat{J}_x,\hat{J}_y\right]=i\hbar\hat{J}_z \qquad \Delta J_x\Delta J_y\geq\left|\frac{1}{2i}\left\langle\left[\hat{J}_x,\hat{J}_y\right]\right\rangle\right|=\frac{\hbar}{2}\left|\left\langle J_z\right\rangle\right|$$

We define raising and lowering operators:

$$\hat{J}_\pm=\hat{J}_x\pm i\hat{J}_y \qquad \left[\hat{J}_z,\hat{J}_\pm\right]=\pm\hbar\hat{J}_\pm$$

From which we derive angular momentum eigenstates and eigenvalues

$$\hat{J}^2\left|j,m\right\rangle=j(j+1)\hbar^2\left|j,m\right\rangle \qquad \hat{J}_\pm\left|j,m\right\rangle=\sqrt{j(j+1)-m(m\pm1)}\left|j,m\pm1\right\rangle$$

$$\hat{J}_z\left|j,m\right\rangle=m\hbar\left|j,m\right\rangle \qquad m=-j,-(j+1),\ldots,(j-1),j$$

Representation of $\hat{J}_x$ and $\hat{J}_y$ for any j are readily
Derived through the representation of the $\hat{J}_\pm$ using

$$\hat{J}_x=\tfrac{1}{2}\left(\hat{J}_++\hat{J}_-\right) \qquad \hat{J}_y=\tfrac{1}{2i}\left(\hat{J}_+-\hat{J}_-\right)$$

The Schrodinger Equation

Time evolution described by unitary transformation of t=0 wave function:

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$$

$$\hat{U}(t) = \exp\left(-i\,\hat{\mathcal{H}}\,t/\hbar\right)$$

From this derive Schrodinger differential equation for t-dependent state

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{\mathcal{H}}|\psi(t)\rangle$$

And Ehrenfest equation for evolution of averages of observables

$$\frac{d}{dt}\langle\hat{A}\rangle = \frac{i}{\hbar}\langle\psi(t)|\left[\hat{\mathcal{H}},\hat{A}\right]\psi(t)\rangle$$

Time required to appreciate change depends on energy spread:

$$\Delta E\Delta t \geq \tfrac{1}{2}\hbar$$

For spin in a field along z-axis $\hat{\mathcal{H}} = -\frac{gq}{2mc}B\hat{S}_z$ and $\hat{U}(t)$ is rot. op.

$$\langle S_x\rangle = \frac{\hbar}{2}\cos\omega t \qquad \langle S_x\rangle = \frac{\hbar}{2}\sin\omega t \qquad \langle S_z\rangle = 0$$

Combined Angular Momentum etc.

Spin pairs can be described using two different sets of basis states:

$\left| j_1, j_2; m_1, m_2 \right\rangle$ or $\left| JM \right\rangle$

$$\left| JM \right\rangle = \sum_{m_1+m_2=M} \left| j_1, j_2; m_1, m_2 \right\rangle \left\langle j_1, j_2; m_1, m_2 \middle| J, M \right\rangle$$

C-G coefficients

$$\left| j_1, j_2; m_1, m_2 \right\rangle = \sum_{J,M=m_1+m_2} \left| JM \right\rangle \left\langle J, M \middle| j_1, j_2; m_1, m_2 \right\rangle$$

Singlet state is rotationally invariant: $\hat{R}(\phi \hat{\mathbf{n}}) \left| 0,0 \right\rangle = \left| 0,0 \right\rangle$

Disintegration of singlet reveals fundamental nature of QM uncertainty

$$\hat{S}_a \longleftarrow \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle \right) \longrightarrow \hat{S}_b$$

Hidden variable theory obeys Bell’s inequality. QM does NOT hence

- QM contains complete information of physical state
- QM is a non-local theory
- QM describes the results of experiments including those probing EPR

Wave mechanics in one dimension

Unitary Translation operator: $\hat{T}(x) = \exp(-i\hat{p}_x x / \hbar)$

$\hat{p}_x$ is momentum operator for x-direction $\hat{p}_x | p_x \rangle = p_x | p_x \rangle$

$[\hat{x}, \hat{p}_x] = i\hbar \quad \Rightarrow \quad \Delta x \Delta p_x \geq \frac{\hbar}{2}$

ENTITY	POSITION SPACE	MOMENTUM SPACE
$\|\psi\rangle$	$\psi(x) = \langle x \| \psi \rangle$	$\Phi(p) = \langle p \| \psi \rangle$
$\hat{p}\|\psi\rangle$	$\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)$	$p\Phi(p)$
$\langle x \| \psi \rangle$	$\psi(x)$	$\frac{1}{\sqrt{2\pi\hbar}}\int \exp(ipx/\hbar)\Phi(p)dp$
$\langle p \| \psi \rangle$	$\frac{1}{\sqrt{2\pi\hbar}}\int \exp(-ipx/\hbar)\psi(x)dx$	$\Phi(p)$

$|\psi(x)|^2$ Is probability of detecting particle between x and x+dx

$|\Phi(p)|^2$ Is probability of measuring momentum between p and p+dp

Time propagation of free particle wave function $\Phi(p,t) = \exp\left(-i\frac{p^2}{2m} t/\hbar\right)\Phi(p,0)$

Scattering and bound states in 1D

Bound states have energy quantization and vanish at infinity
Scattering states have continuous energy distribution and travel to infinity

Bound states occur for $E < V(\pm\infty)$
Scattering states occur for $E > V(\pm\infty)$

If $V(x) > V(\pm\infty)$ there are only scattering states
If $V(x) < V(\pm\infty)$ there can be bound and scattering states

Solve time independent Schrodinger equation:

$$\hat{\mathcal{H}}\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_E(x) + V(x)\psi_E(x) = E\psi_E(x)$$

Solutions must satisfy boundary conditions that can give quantization:

$\psi(0^-) = \psi(0^+)$ Wave function continuous

$-\frac{\hbar^2}{2m}\left(\psi'(0^+) - \psi'(0^-)\right) + \psi(0)\int_{0^-}^{0^+} V(x)dx = 0$ Wave function slope discontinuity reflects any singularity in *V(x)*

Putting in time dependence: $\psi_E(x,t) = \exp(-iEt/\hbar)\psi_E(x,t=0)$

The 1D Harmonic Oscillator

$$\hat{\mathcal{H}}|\psi\rangle = \left(\frac{\hat{p}^2}{2m} + \tfrac{1}{2}m\omega^2\hat{x}^2\right)|\psi\rangle = E|\psi\rangle$$

Can diagonalize by introducing

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$

$$\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

$$\Rightarrow \quad \hat{\mathcal{H}} = \hbar\omega\left(\hat{a}^+\hat{a} + \tfrac{1}{2}\right)$$

Commutation relations determine eigenstates and values:

$$\left[\hat{a}, \hat{a}^+\right] = 1$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

$$\hat{a}^+|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$\Rightarrow$$

$$E_n = \hbar\omega\left(n + \tfrac{1}{2}\right), \quad n = 0,1,2,....$$

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\langle x|\left(\hat{a}^+\right)^n|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(y)e^{-y^2/2}$$

$$y = \sqrt{\frac{m\omega}{\hbar}}x$$

WKB Approximation

An approximation to $\psi(x)$ when $V(x)$ is slowly varying

Define $p^2(x) = 2m\left(E - V(x)\right)$

For classical regime $p^2(x)>0$:

$$\psi(x) \propto \frac{1}{\sqrt{p(x)}} \exp(\pm i \frac{1}{\hbar} \int p(x) dx)$$

For non classical regime $p^2(x)<0$:

$$\psi(x) \propto \frac{1}{\sqrt{|p(x)|}} \exp(\pm \frac{1}{\hbar} \int |p(x)| dx)$$

Quantization conditions for WKB bound states (n=1,2,3,…):

$$\int_{x_1}^{x_2} p(x)dx = \begin{cases} n\pi\hbar & \text{Two hard boundaries} \\ (n-\frac{1}{4})\pi\hbar & \text{One hard boundary} \\ (n-\frac{1}{2})\pi\hbar & \text{No hard boundaries} \end{cases}$$

Good Luck with the Exam!

- Practice exams are available on the course web page
- Problem solving session with George Sunday 12/16 at 6 pm in room 361(usual place)
- Office hours with Collin this afternoon and by appt.
- Final exam on **Tuesday 12/18 9-noon in room 278**
- See you next semester!