QM I Fall 2011

HW #13 Solutions

1) $a|\alpha\rangle = \alpha|\alpha\rangle$

a. $|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$

$c_n = \langle n|\alpha\rangle$

$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle \Rightarrow c_n = \frac{1}{\sqrt{n!}}\langle 0|a^n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\underbrace{\langle 0|\alpha\rangle}_{c_0}$

$\sum_n |c_n|^2 = 1 \Rightarrow |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = 1$

$\therefore |c_0|^2 e^{|\alpha|^2} = 1 \Rightarrow c_0 = e^{-|\alpha|^2/2}$

b. $|\alpha_0(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-iE_nt/\hbar} |n\rangle$

$= \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha_0^n}{\sqrt{n!}} e^{-i(n+\frac{1}{2})\omega t} |n\rangle$

$= e^{-i\omega t/2} \sum_{n=0}^{\infty} \frac{(\alpha_0 e^{-i\omega t})^n}{\sqrt{n!}} e^{-|\alpha_0|^2/2} |n\rangle$

Since the overall phase is irrelevant, and

$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2} |n\rangle$,

$|\alpha(t)\rangle = |\alpha_0 e^{-i\omega t}\rangle$, i.e. $|\alpha(t)\rangle$ is same as $|\alpha\rangle$, except that eigenvalue is time-dependent:

$\alpha(t) = \alpha_0 e^{-i\omega t}$

c) $\langle \alpha_1 | \alpha_2 \rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-|\alpha_1|^2/2} e^{-|\alpha_2|^2/2} \frac{(\alpha_1^*)^n}{\sqrt{n!}} \frac{\alpha_2^m}{\sqrt{m!}} \underbrace{\langle n | m \rangle}_{\delta_{nm}}$

$$= e^{-\frac{|\alpha_1|^2 + |\alpha_2|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha_1^* \alpha_2)^n}{n!}$$

$$= e^{-\frac{1}{2} |\alpha_1 - \alpha_2|^2}$$

$\therefore \langle \alpha_1 | \alpha_2 \rangle \neq 0,$ for $\alpha_1 \neq \alpha_2$

QM I Fall 2011

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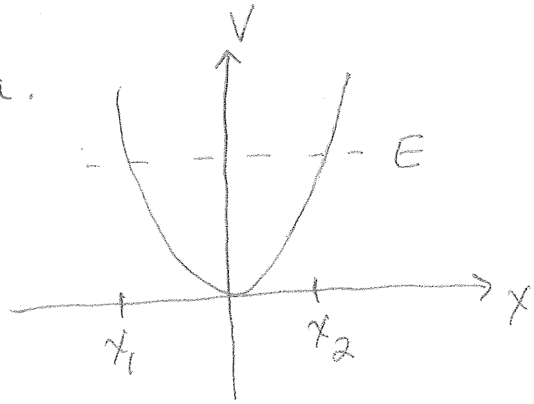
2) a.



$$\int_{x_1}^{x_2} p(x)\,dx = \left(n+\tfrac{1}{2}\right)\pi\hbar, \quad n=0,1,2,\ldots$$

$$p(x) = \sqrt{2m\left(E - \tfrac{1}{2}m\omega^2 x^2\right)}$$

Since $V(x)$ is even, $x_1 = -x_2$

$$\tfrac{1}{2}m\omega^2 x_2^2 = E \Rightarrow x_2 = \frac{1}{\omega}\sqrt{\frac{2E}{m}}$$

$$2m\omega\int_0^{x_2}\sqrt{x_2^2 - x^2}\,dx = \left(n+\tfrac{1}{2}\right)\pi\hbar$$

$$I \equiv \int_0^{x_2}\sqrt{x_2^2-x^2}\,dx = x_2\int_0^{x_2}\sqrt{1-(x/x_2)^2}\,dx$$

$$y \equiv \sin^{-1}\left(\frac{x}{x_2}\right) \Rightarrow \frac{x}{x_2} = \sin y \Rightarrow dx = x_2\cos y\,dy$$

$$I = x_2^2\int_0^{\pi/2}\cos^2 y\,dy = x_2^2\int_0^{\pi/2}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2y\right)dy$$

$$= x_2^2\left[\tfrac{1}{2}y + \tfrac{1}{4}\sin 2y\right]_0^{\pi/2} = \frac{\pi}{4}x_2^2$$

$$\therefore\ 2m\omega\left(\frac{\pi}{4}\right)\left(\frac{1}{\omega^2}\frac{2E}{m}\right) = \left(n+\tfrac{1}{2}\right)\pi\hbar$$

$$\Rightarrow E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega, \quad n=0,1,2,\ldots$$

In this case, WKB approximation produces the exact energies.

QM I Fall 2011

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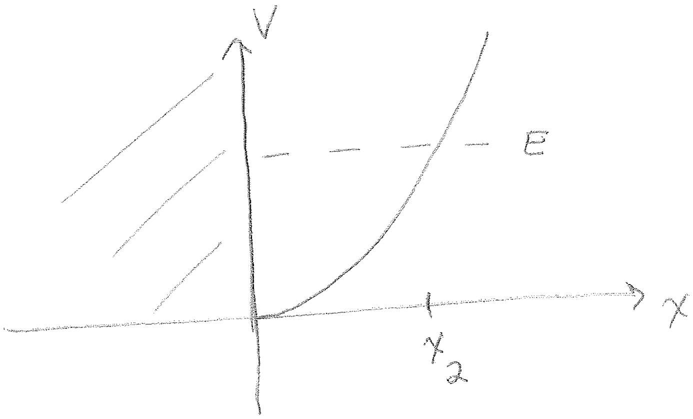
2) b.



Using the connection formulas, find

$$\psi(x) \cong \begin{cases} \dfrac{2A}{\sqrt{p(x)}} \sin\left[\dfrac{1}{\hbar}\displaystyle\int_x^{x_2} p(x')\,dx' + \dfrac{\pi}{4}\right], & x < x_2 \\ \dfrac{A}{\sqrt{|p(x)|}} \exp\left[-\dfrac{1}{\hbar}\displaystyle\int_{x_2}^{x} |p(x')|\,dx'\right], & x > x_2 \end{cases}$$

Boundary condition $\psi(0) = 0$

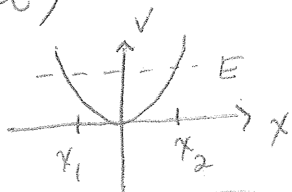
$$\Rightarrow \frac{1}{\hbar}\int_0^{x_2} p(x)\,dx + \frac{\pi}{4} = n\pi, \quad n = 1, 2, 3, \ldots$$

(but not $n=0$ otherwise solution for $x < x_2$ is trivial)

$$\therefore \int_0^{x_2} p(x)\,dx = \left(n - \tfrac{1}{4}\right)\pi\hbar;$$

note this is the WKB quantization condition for a potential in general with one "hard wall." Contrast this with condition for a well with no hard walls: (e.g. the full oscillator potential)

$$\int_{x_1}^{x_2} p(x)\,dx = \left(n + \tfrac{1}{2}\right)\pi\hbar, \quad n = 0, 1, 2, \ldots$$



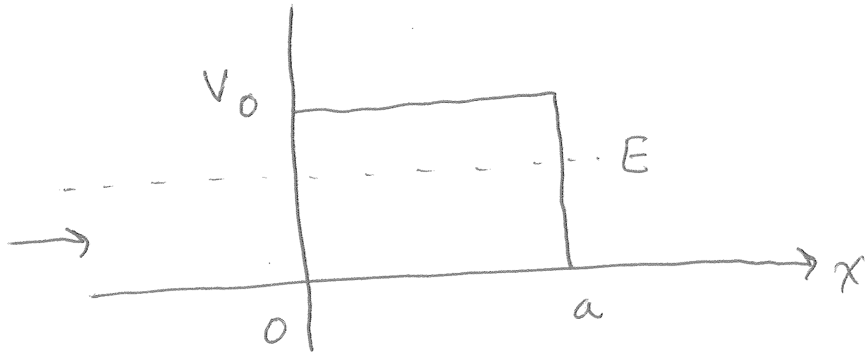
$$p(x) = \sqrt{2m\left(E - \tfrac{1}{2}m\omega^2x^2\right)} = m\omega\sqrt{x_2^2 - x^2}, \quad x_2 = \frac{1}{\omega}\sqrt{\frac{2E}{m}}$$

$$\therefore \int_0^{x_2} p(x)\,dx = m\omega\underbrace{\int_0^{x_2}\sqrt{x_2^2 - x^2}\,dx}_{= \frac{\pi}{4}x_2^2,\text{ as found in HW \#12}} = \frac{\pi E}{2\omega}$$

$$\therefore \frac{\pi E}{2\omega} = \left(n - \tfrac{1}{4}\right)\pi\hbar \Rightarrow E_n = \left(2n - \tfrac{1}{2}\right)\hbar\omega = \left(\tfrac{3}{2}, \tfrac{7}{2}, \tfrac{11}{2}, \ldots\right)\hbar\omega$$

exact energies!

3)



$$T \simeq e^{-2\gamma}$$

$$\gamma = \frac{1}{\hbar}\int_0^a \sqrt{2m(V_0-E)}\,dx = \frac{a}{\hbar}\sqrt{2m(V_0-E)}$$

The exact transmission coefficient is given by Eq. (6.153) of Townsend:

$$T = \frac{1}{1+\left(\frac{k^2+q^2}{2kq}\right)^2 \sinh^2(qa)}$$

$$k = \frac{\sqrt{2mE}}{\hbar}, \quad q = \frac{\sqrt{2m(V_0-E)}}{\hbar} = \frac{\gamma}{a}$$

$$\therefore \quad T = \frac{1}{1+\frac{V_0^2}{4E(V_0-E)}\sinh^2\gamma}$$

$$T \ll 1 \Rightarrow \gamma \text{ large} \Rightarrow \sinh\gamma \approx \frac{1}{2}e^{\gamma}$$

$$\therefore \quad T \approx \frac{1}{1+\frac{V_0^2}{16E(V_0-E)}e^{2\gamma}} \approx \frac{16E(V_0-E)}{V_0^2}e^{-2\gamma}$$

The coefficient out front is of order 1, so the dominant dependence on E is in exponential factor $\Rightarrow T \simeq e^{-2\gamma}$, which is indeed the WKB result.