Solution Midterm

1. [10 points] State the "Fundamental Assumption" of statistical and thermal physics.

The fundamental assumption of statistical mechanics is that a closed system is equally likely to be in any of the quantum states accessible to it.

2. [10 points] A freshly laid egg remains in contact with a heat reservoir (i.e., a hen) until it hatches. During this process, is heat absorbed or given off by the egg? Explain your answer.

We have heat defined as $\delta Q = TdS$ and $\Delta V = 0$ or the work done $W = 0$. Since the temperature is kept constant during the process we have $Q = T\Delta S$. At the beginning the egg is not ordered (the symmetry is close to the one of a liquid) and then the system starts to organize itself into an array of cells which is more ordered (the symmetry is approaching the one of a crystal). Therefore the entropy decrease and $\Delta S < 0$ or $Q < 0$. The heat decreasing in the egg, the heat is transferred from the egg to the hen.

3. A physical model of a rubber band is a chain of N independent segments, each of which can have one of two possible lengths, 0 or λ . If a segment has a length λ , then its energy is zero. If a segment has zero length, then its energy is ϵ. The rubber band is in thermal contact with the atmosphere at temperature T.

(a) [5 points] Find the average length, L, of the rubber band, in terms of N, T, λ, and ϵ.

The partition function for one segment is

$$Z = e^0 + e^{-\beta\epsilon} = 1 + e^{-\beta\epsilon}$$

and the average length for one segment is

$$\langle l \rangle = \frac{\lambda}{1 + e^{-\beta\epsilon}}.$$

For N segments the average is

$$\langle L \rangle = \frac{N\lambda}{1 + e^{-\beta\epsilon}}.$$

(b) [5 points] As the temperature is increased, does the average length increase or decrease?

$$\frac{\partial \langle L \rangle}{\partial \beta} = \frac{N\lambda\beta e^{-\beta\epsilon}}{(1 + e^{-\beta\epsilon})^2}$$

$$\frac{\partial \langle L \rangle}{\partial \beta} > 0 \text{ for } \epsilon > 0$$

$$\frac{\partial \langle L \rangle}{\partial \beta} < 0 \text{ for } \epsilon < 0$$

Therefore when β decrease or T increase $\langle L \rangle$ decrease when $\epsilon > 0$ and increase $\epsilon < 0$

(c) [5 points] What is L in the low temperature limit ($kT \ll \epsilon$)?

Assuming $\epsilon > 0$

$$\lim_{T \to 0} \langle L \rangle = N\lambda$$

(d) [5 points] What is L in the high temperature limit ($kT \gg \epsilon$)?

$$\lim_{T \to 0} \langle L \rangle = \frac{N\lambda}{2}$$

4. Two identical systems S_1 and S_2 are both in thermal contact with a large reservoir and in diffusive contact with one another. For both systems, the free energy, F, is related to the particle number in the same manner: $F_1 = cN_1^2$ and $F_2 = cN_2^2$ where c is a constant (and the same constants for both identical systems).

(a) [15 points] A battery is now put in place that maintains a chemical potential difference

$$\Delta = \mu_{2,ext} - \mu_{1,ext}$$

between the two systems. In diffusive equilibrium, find the number N_1 of particles in S_1 and the number N_2 of particles in S_2, expressed in terms of Δ, c, and the total particle number $N = N_1 + N_2$.

We have

$$\mu_{i,int} = \frac{\partial F_i}{\partial N} = 2cN_i$$

The condition for diffusive equilibrium is

$$\mu_{1,int} + \mu_{1,ext} = \mu_{2,int} + \mu_{2,ext}$$

$$\Rightarrow 2cN_1 = 2c(N - N_1) + \Delta$$

$$\Rightarrow N_1 = \frac{N}{2} + \frac{\Delta}{4c}$$

$$\Rightarrow N_2 = N - N_1 = \frac{N}{2} - \frac{\Delta}{4c}$$

(b) [10 points] Now the battery is disconnected, and useful work is extracted isothermally as the particles flow slowly from S_1 to S_2 until diffusive equilibrium is reestablished. How much work is extracted?

We have $dU = TdS - PdV$ and $F = U - TS$ therefore

$$dF = -SdT - PdV$$

Since the process is isothermal we have $dF = -PdV = \delta W$ or $\Delta F = W$. Therefore the work extracted is $W' = -\Delta F$. We have

$$\Delta F = F_f - F_i = c(N_{1f}^2 + N_{2f}^2) - c(N_{1i}^2 + N_{2i}^2) = c\frac{N^2}{2} - c((\frac{N}{2} + \frac{\Delta}{4c})^2 + (\frac{N}{2} - \frac{\Delta}{4c})^2) = -\frac{\Delta^2}{8c}$$

and

$$W' = \frac{\Delta^2}{8c}$$

5. A container of N atoms of type A low density gas at temperature T and volume V is in thermal equilibrium with an identical container of N atoms of type B of low density gas at temperature T and identical volume V. A constraining wall is removed between the two containers to allow particle diffusion.

(a) [10 points] After equilibrium is achieved, what is the total change of entropy?

The entropy for ideal gas is $S_i = k_b N_i(\ln(n_Q/n_i) + 5/2)$. After the wall removal the density for each gas is divided by 2. The change in entropy is

$$\begin{aligned}\Delta S &= 2k_b N(\ln(\frac{2n_Q}{n}) + 5/2) - 2k_b N(\ln(\frac{n_Q}{n}) + 5/2)\\ &= 2k_b N \ln 2\end{aligned}$$

(I am assuming here same mass for the 2 particles.)

(b) [10 points] What would the total change of entropy be if the type A = type B?

Having a wall or not doesn't change the number of states nor the entropy if the gases are the same.

6. The following questions relate to general properties of a Fermi gas.

(a) [5 points] If the density of a Fermi gas is raised by a factor of a thousand, by what factor does the Fermi energy, ϵ_F, increase or decrease?

The Fermi energy for a $3D$ gas is

$$\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

If $n_f = 1000 n_i$ then $\epsilon_{Ff} = (1000)^{2/3}\epsilon_{Fi} = 100\epsilon_{Fi}$

(b) [5 points] If the temperature of a Fermi gas is raised by a factor of two, by what factor does the Fermi energy, ϵ_F, increase or decrease?

Independent of the temperature.

(c) [5 points] Sketch a qualitative graph that illustrates the general behavior of $f(\epsilon)$ vs. ϵ for a Fermi gas. Include a curve for absolute zero temperature, and two slightly higher temperatures to indicate the nature of the trend.

See figure 7.5 in Kittel.