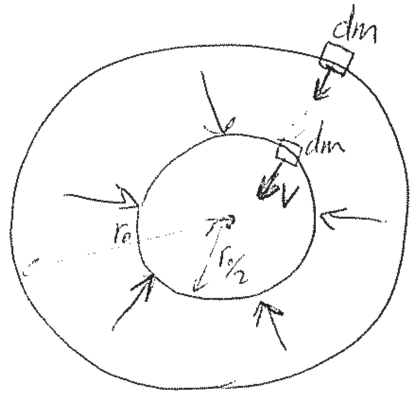
STELLAR PHYSICS
MIDTERM
Solutions

1.(a) Free-fall collapse time, t_{ff}, is the time that it takes for a cloud of density ρ to collapse when there is no pressure gradient to oppose gravitational collapse.



(b) We apply conservation of energy to a mass element dm on the surface of the cloud as it contracts to half it's initial radius (r_0). If V is the velocity of this mass element at $r_0/2$, we have:

$$-\frac{GM\,dm}{r_0} = -\frac{GM\,dm}{r_0/2} + \frac{1}{2}\,dm\,V^2$$

$$\Rightarrow V^2 = \frac{2GM}{r_0} \Rightarrow V = \sqrt{\frac{2GM}{r_0}}$$

We can approximate the average speed of this mass element throughout this contraction to be $V/2$

So, the time of this collapse is about

$$t_{1/2} = \frac{r_0/2}{V/2} = \frac{r_0}{V} = \frac{r_0}{\sqrt{\frac{GM}{r_0}2}} = \sqrt{\frac{1}{2G\frac{M}{r_0^3}}} \Rightarrow t_{1/2} = \sqrt{\frac{3}{8\pi G\rho}}$$

$$\frac{M}{\frac{4\pi r_0^3}{3}} = \rho \Rightarrow \frac{M}{r_0^3} = \frac{4\pi\rho}{3}$$

Note that $t_{1/2}$, is more than half of t_{ff}, because the collapse is going to be much faster after the cloud reaches half it's initial radius. So, ignoring dimensionless constants, free-fall collapse time is about:

$$t_{ff} \approx \sqrt{\frac{1}{G\rho}}$$

(c) A cloud of radius R will ~~condense~~ contract under own self gravity, if it's mass exceeds 'Jeans mass'.

(d) Jeans length is where sound-crossing time is the same as free-fall time.

$$t_{sc} \sim t_{ff} \Rightarrow \frac{R_J}{c_s} \sim \frac{1}{\sqrt{G\rho}} \Rightarrow \left.\begin{aligned} & R_J \sim \frac{c_s}{\sqrt{G\rho}} \\ & c_s = \sqrt{\frac{\gamma P}{\rho}} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow R_J \sim \frac{\gamma^{1/2} P^{1/2}}{\rho^{1/2}} \frac{1}{G^{1/2}\rho^{1/2}} \sim \frac{\gamma^{1/2}}{\sqrt{G}} \frac{P^{1/2}}{\rho}$$

Jeans mass is the mass inside a sphere of radius R_J.

$$M_J \sim \frac{4\pi}{3} \rho R_J^3 \sim \frac{4\pi}{3} \rho \frac{\gamma^{3/2}}{G^{3/2}} \frac{P^{3/2}}{\rho^{3}}$$

$$\Rightarrow \boxed{M_J \propto \rho^{-2} P^{3/2}}$$

2. (a)

Assuming the sun to be an ideal gas cloud, the total thermal energy is $\frac{3}{2}Nk_BT = \frac{3}{2}\frac{M_\odot}{\bar{m}}k_B T_{\odot,int}$

$$\Rightarrow E_{th} \approx \frac{3}{2} \times \frac{2\times10^{30}}{0.7\times1.67\times10^{-27}} \times 1.38\times10^{-23} \times 6\times10^{6} = 2.1\times10^{41}\,J$$

If the sun shines with the same luminosity as today the time for it to emit as much as its thermal energy is

$$t_{th} = \frac{E_{th}}{L_\odot} = \frac{2.1\times10^{41}\,J}{4\times10^{26}\,W} = 5.3\times10^{14}\ \text{seconds} \approx 1.7\times10^{7}\ \text{years}$$

Total gravitational energy of the sun is about.

$$E_{GR} \approx \frac{GM_\odot^2}{R_\odot} = \frac{6.67\times10^{-11}\times(2\times10^{30})^2}{7\times10^{8}} = 3.8\times10^{41}$$

If the sun emits energy solely from it's gravitational collapse energy, the time it takes to finish its gravitational energy with the current luminosity is:

$$t_{GR} = \frac{E_{GR}}{L_\odot} = \frac{3.8 \times 10^{41}\ J}{4 \times 10^{26}\ W} = 9.5 \times 10^{14}\ \text{second} \approx t_{thermal} \approx 3 \times 10^{7}\ \text{years}$$

(It's because of virial theorem)

Notice that the number we got is much smaller than the known age of the earth $\approx 4.5 \times 10^{9}$ years

(b) The mass fraction of the sun to be burned is $f . M_\odot$. So, based on the given efficiency, the nuclear energy released is:

$$E_{Nu} = 0.005 \times f \times M_\odot \times c^2 = f . 9 \times 10^{44}\ J$$

The time it takes to emit all this energy with the current luminosity is:

$$t_{Nu} = \frac{E_{Nu}}{L_\odot} = \frac{f . 9 \times 10^{44}\ J}{4 \times 10^{26}\ W} = 2.3 \times 10^{18} f .\ \text{seconds}$$

$$= 7.3 \times 10^{10} f\ \cancel{\text{seconds}}\ \text{years}$$

Even with f of about 10% it gives a time comparable to the age of the earth $\sim 4.5 \times 10^{9}$ years.

(c) f must be a very small number, because nuclear burning only happens in the core of the sun, which is a small fraction of the sun.

(d) Convection, Conduction of e^- (but insufficient in normal stars)

(e)

$$\left.\begin{array}{l} k = \dfrac{\sigma}{\bar{m}} \\ \\ l = \dfrac{1}{n\sigma} \end{array}\right\} \rightarrow l = \frac{1}{n\bar{m}k} = \frac{1}{k\rho}$$

$$\rho = n\bar{m}$$

3. (a) Pressure is force per unit area or energy per unit volume. At hydrostatic equilibrium pressure of the gas is proportional to its gravitational potential energy density, so it can balance gravity (virial theorem)

$$P \propto \frac{E_{GR}}{V} \approx \frac{\frac{GM^2}{R}}{R^3}$$

$$\Rightarrow \boxed{P \propto \frac{M^2}{R^4}}$$

(b)

$$\left.\begin{array}{l} P = K_{NR}\underset{\uparrow \, \rho \propto \frac{M}{R^3}}{\rho^{5/3}} \propto \frac{M^{5/3}}{R^5} \\ \text{from (a)} \rightarrow P \propto \frac{M^2}{R^4} \end{array}\right\} \rightarrow \frac{M^{5/3}}{R^5} \propto \frac{M^2}{R^4} \Rightarrow \boxed{M \propto R^{-3}} \text{ as } M\uparrow, R\downarrow$$

(c)

$$\left.\begin{array}{l} P = K_R \rho^{4/3} \propto \frac{M^{4/3}}{R^4} \\ \text{from (a)} \rightarrow P \propto \frac{M^2}{R^4} \end{array}\right\} \rightarrow \frac{M^{4/3}}{R^4} \propto \frac{M^2}{R^4}$$

$\Rightarrow$ mass does not scale with radius

$\rightarrow$ Defines Chanrasakhar mass, maximum mass that can be stablized by degenrate e^- $\rightarrow$ $\boxed{\text{max mass of WD}}$