STELLAR PHYSICS

Homework 1 – Solution

1.

$$\rho = the\ mean\ density\ of\ the\ universe\ at\ the\ present\ time = 5 \times 10^{-27} \frac{Kg}{m^3}$$

$$t_{FF} = the\ free - fall\ collapse\ time\ of\ a\ cloud\ of\ density\ \rho \sim \sqrt{\frac{3\pi}{32G\rho}}$$

$$= \sqrt{\frac{3\pi}{32 \times 5 \times 10^{-27} Kg.m^{-3} \times 6.67 \times 10^{-11} m^3 Kg^{-1} s^{-2}}} = \sqrt{8.8 \times 10^{35} s^2}$$

$$= 9.4 \times 10^{17} s = 3.0 \times 10^{10} years$$

Note that the Friedman equation for the evolution of a flat, matter-dominated (no dark energy) universe is:

$$H^2 = \frac{8\pi G}{3} \rho$$

This is derived from the more complicated theory of General Relativity. The reciprocal of H_0 (the Hubble constant [!] at present time) is a fair estimate of the age of the universe.

$$t = age\ of\ the\ universe \approx \sqrt{\frac{3}{8\pi G\rho}}$$

This is only $\frac{2}{\pi}$ times the free-fall collapse time and is 1.9×10^{10}years. As you see, when matter dominates, gravity dominates, and you get a useful timescale from the free-fall collapse time.

2.

(a)

$$E_{GR} = gravitational\ potential\ energy = \int_0^M \frac{Gm(r)}{r} dm = \int_0^R \frac{Gm(r)}{r} \rho(r) 4\pi r^2 dr$$

We know that

$$m(r) = \int_0^r \rho(r') 4\pi r'^2 dr' = \frac{4}{3}\pi r^3 \rho$$

So,

$$\Rightarrow E_{GR} = -\frac{16\pi^2}{3} G\rho^2 \int_0^R r^4 dr = -\frac{16\pi^2}{3} G\rho^2 \frac{R^5}{5}$$

$$= -\frac{3}{5} \frac{G\left(\frac{4\pi}{3} R^3 \rho\right)^2}{R} = -\frac{3}{5} \frac{GM^2}{R}$$

Average internal pressure is given by the Virial Theorem.

$$< P >= -\frac{1}{3}\frac{E_{GR}}{V} = \frac{1}{3}\frac{3}{5}\frac{GM^2}{R}\frac{3}{4\pi R^3} = \frac{3}{20\pi}\frac{GM^2}{R^4}$$

(b)

If $\rho(r) = \rho_C(1 - {}^{r}/_{R})$, $m(r)$ and E_{GR} will change as follows.

$$m(r) = \int_0^r \rho(r')4\pi r'^2 dr' = \rho_C\, 4\pi \int_0^r (1 - {}^{r'}/_{R}) r'^2 dr'$$

$$= \rho_C 4\pi \int_0^r \left(r'^2 - {}^{r'^3}/_{R}\right) dr' = \rho_C 4\pi \left(\frac{r^3}{3} - \frac{r^4}{4R}\right)$$

$$\Rightarrow E_{GR} = -(4\pi)^2 G{\rho_C}^2 \int_0^R \frac{\left(\frac{r^3}{3} - \frac{r^4}{4R}\right)}{r} (1 - \frac{r}{R}) r^2 dr$$

$$= -16\pi^2 G{\rho_C}^2 \int_0^R (\frac{r^6}{4R^2} - \frac{r^5}{3R} - \frac{r^5}{4R} + \frac{r^4}{3}) dr$$

$$= -16\pi^2 G{\rho_C}^2 \left(\frac{R^5}{28} - \frac{7R^5}{72} + \frac{R^5}{12}\right) = -16\pi^2 G{\rho_C}^2 \left(\frac{11R^5}{504}\right)$$

Noting $M = \rho_C \frac{4\pi}{12} R^3$, we can rewrite it as follows.

$$E_{GR} = -\frac{22}{7}\frac{G\left(\frac{4\pi}{12}R^3\rho_C\right)^2}{R} = -\frac{22}{7}\frac{GM^2}{R}$$

And average internal pressure,

$$< P >= -\frac{1}{3}\frac{E_{GR}}{V} = \frac{1}{3}\frac{22}{7}\frac{GM^2}{R}\frac{3}{4\pi R^3} = \frac{22}{28\pi}\frac{GM^2}{R^4}$$

3.

First, note that there is a typo in the homework question. $10^{-27} Kg\ m^{-3}$ is the approximate density of the universe at the present time. The density of the universe at the recombination era was 1200^3 times more than today's value. (You'll probably learn the reason later in the course) So the density of the primordial cloud in question is about $1.7 \times 10^{-27+9} Kg\ m^{-3} = 1.7 \times 10^{-18} Kg\ m^{-3}$.

Jeans criterion says that the cluster cloud would have been able to collapse if the density of the universe was more than Jeans density. ($\rho > \rho_J$)

$$\rho_J = \frac{3}{4\pi M^2}\left[\frac{3kT}{2G\overline{m}}\right]^3$$

$$M = mass\ of\ the\ cloud = \#\ of\ stars \times average\ mass\ of\ stars = 5 \times 10^5 \times 0.5 \times M_\odot$$
$$= 2.5 \times 10^5 M_\odot = 5.0 \times 10^{35} Kg$$

$$T \approx 10^4 K$$

We need to know the average mass of the particles. ($\overline{m}$) We know that at the time, about 25% of the matter was in form of Helium 4 and the rest was mostly Hydrogen. The mass density of Hydrogen atoms was 3 times the mass density of Helium atoms. Since the mass of a Helium atom is 4 times the mass of a Hydrogen atom, the number density of Hydrogen atoms was 12 times more than number density of Helium atoms. From this we can calculate the average mass of the particles in the primordial cloud. (Note that we've assumed that there is no free electron left. All of the electrons are combined with the nuclei.)

$$\overline{m} = \frac{12 \times 1\ a.m.u. + 1 \times 4\ a.m.u.}{13} = 1.23\ a.m.u. = 1.23 \times 1.66 * 10^{-27} Kg$$
$$= 2.04 \times 10^{-27} Kg$$

$$\Rightarrow \rho_J = \frac{3}{4\pi (5.0 \times 10^{35})^2} \left[\frac{3 \times 1.38 \times\ 10^{-23} \times 10^4}{2 \times 6.67 \times 10^{-11} \times 2.04 \times 10^{-27}} \right]^3 = 3.4 \times 10^{-18} Kg\ m^{-3}$$

So, the density of the universe was at the same order of magnitude with Jeans density at the time the cloud was collapsing. ($\rho \approx 10^{-18} Kg\ m^{-3} \sim \rho_J$) And, M13 could have been formed just after the recombination era.

4.

Equation (1.9) reads

$$P = \frac{2}{3} n < \frac{1}{2} m v^2 >$$

The average translational kinetic energy is the sum of the average translational kinetic energy associated with all of the three dimensions.

$$< \frac{1}{2} m v^2 > = < \frac{1}{2} m ({v_x}^2 + {v_y}^2 + {v_z}^2) > = < \frac{1}{2} m {v_x}^2 > + < \frac{1}{2} m {v_y}^2 > + < \frac{1}{2} m {v_z}^2 >$$

According to equipartition theorem average kinetic energy associated with each dimension is $\frac{1}{2} k_B T$. So the average of total translational kinetic energy is $\frac{3}{2} k_B T$.

$$\Rightarrow P = \frac{2}{3} n \frac{3}{2} k_B T = n k_B T$$

Or in probably in more familiar form

$$PV = N k_B T$$

It's the equation of state for an ideal gas.