STELLAR PHYSICS

Homework 10

Solutions

1.

5.3

We assume that for any member of this family of stars the density at distance r from the center can be written as follows. (just like problem 5.2 of last homework)

$$\rho(r) = \frac{M}{R^3} F_\rho(x)$$

Here function $F_\rho(x)$ is common to the entire family. Also, we assume that the mass enclosed by a sphere of radius r can be written as follows.

$$m(r) = M F_m(x)$$

Here function $F_m(x)$ is also common to the entire family of stars.

Just like problem 5.2 of last homework, assuming the material of star ia classical ideal gas, we find,

$$P(r) = \frac{M^2}{R^4} F_P(x)\,, T(r) = \frac{M}{R} F_T(x)\,.$$

$F_P(x)$ and $F_T(x)$ are again common functions among family members.

The difference between this problem and the 5.2 is the opacity and the energy generation. So, we expect to get different relations for L_{rad} and L_{fus} from problem 5.2. At outer parts of the star where the heat transfer is mostly done radiative diffusion, we have,

$$\frac{dT(r)}{dr} = -\frac{3}{4ac}\frac{\kappa(r)\,\rho(r)}{|T(r)|^3}\frac{L_{rad}(r)}{4\pi r^2}$$

$$\Rightarrow \frac{1}{R}\frac{dT(r)}{dx} = -\frac{3}{4ac}\frac{\kappa_0 \frac{M}{R^3} F_\rho(x)}{\left(\frac{M^3}{R^3}\right) {F_T}^3(x)}\frac{L_{rad}(r)}{4\pi R^2\, x^2}$$

$$\Rightarrow \frac{1}{R}\frac{M}{R}\acute{F}_T(x) = -\frac{3}{4ac}\frac{\kappa_0 \frac{M}{R^3} F_\rho(x)}{\left(\frac{M^3}{R^3}\right) {F_T}^3(x)}\frac{L_{rad}(r)}{4\pi R^2\, x^2}$$

$$\Rightarrow L_{rad}(r) = M^3 \frac{16\pi ac}{3\kappa_0}\frac{\acute{F}_T(x){F_T}^3(x)x^2}{F_\rho(x)} = \boxed{M^3\, F_{rad}(x)}$$

Here $F_{rad}(x)$is common among family members. Assuming CNO cycle domination throughout the star, we have,

$$\frac{dL_{fus}(r)}{dr} = 4\pi r^2 \varepsilon(r)$$

$$\Rightarrow \frac{1}{R}\frac{dL_{fus}(r)}{dx} = 4\pi R^2 x^2 \varepsilon_0 \rho^2(r)\, T^{18}(r)$$

$$\Rightarrow \; dL_{fus}(r) = 4\pi\varepsilon_0 \, R^3 \, x^2 \left(\frac{M^2}{R^6}\right) F_\rho{}^2(x) \left(\frac{M^{18}}{R^{18}}\right) F_T{}^{18}(x) \; dx$$

$$\Rightarrow L_{fus}(r) = \frac{M^{20}}{R^{21}} 4\pi\epsilon_0 \int_0^x x'^2 F_\rho^2(x') F_T^{18}(x') dx' = \frac{M^{20}}{R^{21}} F_{fus}(x)$$

Here $F_{fus}(x)$is also common among stars in this family.

For all of the stars in this family the $L_{rad}(r)$ must be the same as $L_{fus}(r)$ at any distance r from the center. (Except for convective zones, where equation of radiative heat transfer does not hold.) This is also true at the surface of the star where $r = R$ and $x = 1$.

$$L_{rad}(R) = L_{fus}(R)$$

$$\Rightarrow M^3 \, F_{rad}(1) = \frac{M^{20}}{R^{21}} \, F_{fus}(1)$$

$$\Rightarrow R^{21} \propto M^{17} \Rightarrow \boxed{R \propto M^{\frac{17}{21}}}$$

$$\boxed{L(R) \propto M^3}$$

To get the relation between L and the effective temperature (T_E) we remind that the effective temperature is defined by the blackbody formula.

$$Flux = \frac{L}{4\pi R^2} = \sigma T_E^4 \; \Rightarrow L = 4\pi\sigma R^2 T_E^4$$

$$\Rightarrow T_E \propto \frac{L^{\frac{1}{4}}}{R^{\frac{1}{2}}}$$

From the two boxed equations above we have $M \propto R^{\frac{21}{17}} \Rightarrow L \propto R^{\frac{63}{17}} \Rightarrow R \propto L^{\frac{17}{63}} \Rightarrow R^{\frac{1}{2}} \propto L^{\frac{17}{126}}$.

$$\Rightarrow T_E \propto \frac{L^{\frac{1}{4}}}{L^{\frac{17}{126}}} = L^{\frac{29}{252}}$$

$$\Rightarrow \boxed{L \propto T_E^{\frac{252}{29}}}$$

So, on the HR diagram, these stars are on a line with slope of $\frac{252}{29} = 8.69$.

2.

5.4

$$P_c < \left[\frac{\pi}{6}\right]^{1/3} G\ M^{2/3} {\rho_c}^{4/3}$$

$$\beta\ P_c = classical\ ideal\ gas\ pressure = n_c\ k_B T_c = \frac{\rho_c}{\overline{m}}\ k_B T_c$$

$$(1-\beta)P_c = radiation\ pressure = \frac{1}{3} a {T_c}^4$$

We take the fourth power of classical ideal gas pressure line, eliminate ${T_c}^4$ using the radiation pressure line, and solve for P_c.

$$\beta^4\ {P_c}^4 = \frac{{\rho_c}^4}{\overline{m}^4}\ {k_B}^4 {T_c}^4$$

$${T_c}^4 = \frac{3}{a}(1-\beta)P_c$$

$$\Rightarrow \beta^4\ {P_c}^4 = \frac{{\rho_c}^4}{\overline{m}^4}\ {k_B}^4 \frac{3}{a}(1-\beta)P_c$$

$$\Rightarrow {P_c}^3 = \frac{3\ {\rho_c}^4\ {k_B}^4}{\overline{m}^4\ a} \frac{(1-\beta)}{\beta^4}$$

On the other hand, taking the third power of the inequality, we have,

$${P_c}^3 < \frac{\pi}{6} G^3\ M^2 {\rho_c}^4.$$

We replace ${P_c}^3$into the inequality.

$$\Rightarrow \frac{3\ {\rho_c}^4\ {k_B}^4}{\overline{m}^4\ a} \frac{(1-\beta)}{\beta^4} < \frac{\pi}{6} G^3\ M^2 {\rho_c}^4$$

$$\Rightarrow \boxed{\frac{(1-\beta)}{\beta^4} < \frac{\pi}{18} \frac{G^3\ \overline{m}^4\ a}{{k_B}^4} M^2}$$

$$\Rightarrow \frac{(1-\beta)}{\beta^4} < 2.029 \times 10^{-3} \left(\frac{M}{M_\odot}\right)^2$$

We can now numerically solve this inequality for the fraction of pressure due to radiation at the center of the star. $(1-\beta)$ So, for different values of M we have:

$$M = M_{\odot} \xRightarrow{Mathematica} (1-\beta) < 2.013 \times 10^{-3}$$

$$M = 4\ M_{\odot} \xRightarrow{Mathematica} (1-\beta) < 2.887 \times 10^{-2}$$

$$M = 40\ M_{\odot} \xRightarrow{Mathematica} (1-\beta) < 4.055$$