STELLAR PHYSICS
Homework 9
Solutions

1.

The graph in the hand out shows how rate of energy released per unit mass changes with temperture in p-p chain and CNO cycle. You can see at the temperture of solar core $\sim 10^7$K p-p chain dominates.

2.(5.1)

$$\frac{dP}{dr} = -\frac{4\pi}{3} G \rho_c^2 r \exp(-r^2/a^2)$$

The pressure inside the star is obtained by integrating this equation and impose the boundry condition of zero pressure at $r = R$. (See page 151 of Phillips book)

$$P(r) = \frac{2\pi}{3} G \rho_c^2 a^2 \left[\exp(-r^2/a^2) - \exp(-R^2/a^2) \right]$$

From virial theorem we know

$$\langle P \rangle = -\frac{E_{GR}}{3V} \Rightarrow \boxed{E_{GR} = -3V \langle P \rangle}$$

So, for obtaining the gravitational potential energy we need to calculate $\langle P \rangle$, the average pressure inside the star.

Average

We calculate volume averge of pressure as follows,

$$\langle P\rangle = \frac{\int P(r)\,d^3V}{\int d^3V} = \frac{4\pi r}{\underset{\frac{4\pi R^3}{3}}{V}}\int_0^R r^2 P(r)\,dr$$

$$= \frac{3}{R^3}\cdot\frac{2\pi}{3}G\rho_c^2 a^2 \int_0^R \left[r^2\exp\left(-\frac{r^2}{a^2}\right) - r^2\exp\left(-\frac{R^2}{a^2}\right)\right]dr$$

$$= \frac{2\pi}{3}G\rho_c^2 a^2\left[\frac{3a^3}{R^3}\int_0^{R/a} x^2\exp(-x^2)\,dx - \exp\left(-\frac{R^2}{a^2}\right)\right]$$

$$= \frac{2\pi}{3}G\rho_c^2 a^2\left[\frac{3a^3}{R^3}\left[\frac{-1}{2}\frac{R}{a}\exp\left(-\frac{R^2}{a^2}\right)+\frac{\sqrt{\pi}}{2}\underset{\text{(error function)}}{\mathrm{erf}}\left(\frac{R}{a}\right)\right]-\exp\left(-\frac{R^2}{a^2}\right)\right]$$

$$\Rightarrow E_{GR} = -3V\langle P\rangle =$$

$$= -\cancel{3}\,\frac{4\pi}{\cancel{3}}R^3\,\frac{2\pi}{3}G\rho_c^2 a^2\left[-\frac{3a^2}{2R^2}\exp\left(-\frac{R^2}{a^2}\right) - \exp\left(-\frac{R^2}{a^2}\right) + \frac{3\sqrt{\pi}a^3}{2R^3}\mathrm{erf}\left(\frac{R}{a}\right)\right]$$

$$\boxed{= \frac{8\pi^2}{3}G\rho_c^2 a^5\left[-\frac{3R}{2a}\exp\left(-\frac{R^2}{a^2}\right) - \frac{R^3}{a^3}\exp\left(-\frac{R^2}{a^2}\right) + \frac{3\sqrt{\pi}}{2}\mathrm{erf}\left(\frac{R}{a}\right)\right]}$$

If $R >> a$ the first two terms involving $\exp(-R^2/a^2)$ are negligible. "error function" defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

is close to ~~unity~~ +1 when the argument x is a large ~~negative~~ number. So, $\mathrm{erf}(R/a) \approx 1$

$$\Rightarrow \boxed{E_{GR} \approx \frac{2\cancel{4}\pi^2}{\cancel{3}} \frac{\cancel{3}\sqrt{\pi}}{\cancel{2}} G \rho_c^2 a^5 = 2\pi^2 \sqrt{\pi} G \rho_c^2 a^5}$$

As it's been done in Phillips book (pages 151-152) we can solve for $m(r)$. (eq. 5.26)

$$m(r) = \frac{4\pi a^3}{3} \rho_c \Phi(x), \quad \text{where } x = \frac{r}{a}$$

$$\Rightarrow m(R) = M = \frac{4\pi a^3}{3} \rho_c \Phi(R/a)$$

And, we have

$$\Phi^2(R/a) = 6 - 3\left(\frac{R^4}{a^4} + 2\frac{R^2}{a^2} + 2\right) \exp(-R^2/a^2)$$

All terms involving $\exp(-R^2/a^2)$ are negligible when $R >> a$

$$\Rightarrow \Phi(R/a) \approx \sqrt{6}$$

$$\Rightarrow M = \frac{4\pi a^3}{3} \rho_c \sqrt{6}$$

$$\Rightarrow \rho_c = \cancel{\frac{3}{4\pi}} \frac{\sqrt{3}\, M}{4\sqrt{2}\,\pi a^3}$$

We replace ρ_c into the expression we got for E_{GR}

$$E_{GR} \approx 2\pi^2 \sqrt{\pi}\, G \left(\frac{\sqrt{3}\, M}{4\sqrt{2}\,\pi a^3}\right)^2 a^5$$

$$\boxed{\approx \frac{3\sqrt{\pi}}{16} \frac{GM^2}{a} \approx \frac{1}{3} \frac{R}{a} \frac{GM^2}{R}}$$

3. (5.2)

We know $\rho(r) = \frac{M}{R^3} F_\rho(\overset{r/R}{x})$, and $m(r) = M F_m(x)$.

From hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{G\, m(r)\, \rho(r)}{r^2}$$

$$= -\frac{GM}{R^3} \cdot M \cdot \frac{F_\rho(x)\, F_m(x)}{r^2}$$

$$= -\frac{GM^2}{R^3} \frac{1}{R^2} \frac{F_\rho(x) \cdot F_m(x)}{x^2}$$

$$\Rightarrow dP = -\frac{GM^2}{R^5} \frac{F_\rho(x)\, F_m(x)}{x^2}\, dr$$

$$= \frac{-GM^2}{R^4} \frac{F_\rho(x)\, F_m(x)}{x^2}\, dx$$

Assuming the boundry condition $P = 0$ at $r = R$ we have:

$$P(r) = \frac{-GM^2}{R^4} \underbrace{\int_1^x \frac{F_\rho(x')\, F_m(x')}{x'^2}\, dx'}_{\to F_p(x)} = \boxed{\frac{M^2}{R^4} F_p(x)}$$

Assuming the star is ideal classical gas we have.

$$P(r) = n(r) k_B T(r) = \frac{\rho(r)}{\overline{m}} k_B T(r)$$

Since we assumed the star is chemically homogeneous, $\bar{m}$ does not depend on r

$$\Rightarrow T(r) = \frac{\bar{m}}{k_B}\frac{P(r)}{\rho(r)}$$

$$= \frac{\bar{m}}{k_B}\frac{\frac{M^2}{R^4}}{\frac{M}{R^3}}\frac{F_P(x)}{F_\rho(x)} = \frac{M}{R}\overbrace{\frac{\bar{m}F_P(x)}{k_B F_\rho(x)}}^{F_T(x)} = \boxed{\frac{M}{R}F_T(x)}$$

Opacity is described by Kramers' Law.

$$\kappa \propto \rho T^{-3.5} \Rightarrow \kappa(r) = C\rho(r)T(r)^{-3.5}$$

If energy transport is by radiative diffusion we have

$$\frac{dT}{dr} = \frac{-3}{4ac}\frac{\kappa(r)\rho(r)}{T(r)^3}\frac{L(r)}{4\pi r^2}$$

$$\Rightarrow \frac{1}{R}\frac{dT}{dx} = \frac{-3}{4ac}\frac{C\rho(r)T(r)^{-3.5}\rho(r)}{T(r)^3}\frac{L(r)}{4\pi x^2}\frac{1}{R^2}$$

$$\Rightarrow \frac{M}{\cancel{R}}F_T'(x) = \frac{-3}{4ac}\frac{C\left(\frac{M}{R^3}\right)^2 F_\rho^2(x)}{\left(\frac{M}{R}\right)^{6.5}F_T^{6.5}(x)}\frac{L(r)}{4\pi x^2}\frac{1}{R}$$

$$\Rightarrow L_{rad}(r) = \frac{M^{5.5}}{R^{0.5}} \overbrace{\frac{16\pi ac}{3G} \frac{F_T'(x) F_T^{6.5}(x) \cdot x^2}{F_\rho^2(x)}}^{F_{rad}(x)}$$

$$\boxed{= \frac{M^{5.5}}{R^{0.5}} F_{rad}(x)}$$

Accordind to (5.4)

$$\frac{dL}{dr} = 4\pi r^2 \mathcal{E}(r),$$

and $$\mathcal{E}(r)_{pp} = \underbrace{9.5 \times 10^{-37} X_1^2}_{A} \rho^2(r)\, T^4(r)\ W m^{-3}$$

$$\Rightarrow \quad dL = 4\pi r^2 \mathcal{E}(r)\, dr$$

$$= 4\pi R^2 x^2 A \rho(r)^2 T^4(r)\ R\, dx$$

$$= 4\pi A\ R^3 \left(\frac{M}{R^3}\right)^2 \left(\frac{M}{R}\right)^4 x^2 F_\rho^2(x) F_T^4(x)\, dx$$

$$= \frac{M^6}{R^7} 4\pi A x^2 F_\rho^2(x) F_T^4(x)\, dx$$

~~Assu~~ Imposing $L=0$ at $r=0$ we have

$$L_{fus} = \frac{M^6}{R^7} \underbrace{\int_0^x 4\pi A x'^2 F_\rho^2(x') F_T^4(x')\, dx'}_{F_{fus}(x)}$$

$$\boxed{\Rightarrow L_{fus} = \frac{M^6}{R^7} F_{fus}(x)}$$