Homework 10.

1) Problem 4.8

Fundamental equation of the Simple Model is (4.15)

$$Z(t) = Z(t=0) + p \ln\left(\frac{M_g(t=0)}{M_g(t)}\right)$$

Usually introduce the gas fraction

$$\mu(t) = \frac{M_g(t)}{M_{tot}} = \frac{M_g(t)}{M_g(t=0)} = \frac{M_g(t)}{M_*(t)+M_g(t)}$$

$$\Rightarrow \quad Z(t) = Z(t=0) + p \ln(1/\mu)$$

For $Z(t=0) = 0$

$$\frac{Z(t)}{p} = \ln(1/\mu) \equiv \frac{Z(\mu)}{p}$$

This is the metallicity of stars forming at the time when the gas fraction is μ. If have long-lived stars, then their mean metallicity

$$\langle Z_* \rangle = \frac{1}{M_*}\int_0^{M_*} Z(M_*')\,dM_*'$$

We have

$$\frac{Z}{p} = \ln(1/\mu)$$ and need to rewrite in terms of M_*

$$\mu = \frac{M_g}{M_g + M_*} \qquad 1 - \mu = \frac{M_*}{M_g + M_*} \qquad M_g + M_* = M_{TOT}$$

$$\frac{Z}{p} = \ln\left(\frac{M_{TOT}}{M_{TOT} - M_*}\right)$$

$$= \ln M_{TOT} - \ln(M_{TOT} - M_*).$$

Hence

$$\frac{\langle Z_* \rangle}{p} = \frac{1}{M_*}\int_0^{M_*} dM'_* \left\{\ln M_{TOT} - \ln(M_{TOT} - M'_*)\right\}$$

$$= \ln M_{TOT} - \frac{1}{M_*}\int_0^{M_*} dM'_* \ln(M_{TOT} - M'_*)$$

Substitute $\quad y = M_{TOT} - M'_*$

$$dy = -dM'_*$$

$$\int_0^{M_*} dM_*' \ln(M_{TOT} - M_*')$$

$$= -\int_{M_{TOT}}^{M_{TOT}-M_*} dy \ln y$$

$\int \ln x \, dx = x \ln x - x.$

$$= (y - y \ln y)\Big|_{M_{TOT}}^{M_{TOT}-M_*}$$

$$= M_{TOT} - M_* - (M_{TOT} - M_*) \ln(M_{TOT} - M_*)$$
$$- M_{TOT} + M_{TOT} \ln M_{TOT}$$

$$= -M_* - M_{TOT} \left\{ \ln(M_{TOT} - M_*) - \ln M_{TOT} \right\}$$
$$+ M_* \left(\ln(M_{TOT} - M_*) \right)$$

$$= -M_* + M_* \ln(M_{TOT} - M_*) + M_{TOT} \ln\left(\frac{M_{TOT}}{M_{TOT} - M_*} \right)$$

$$\frac{\langle Z_*\rangle}{p} = \ln M_{TOT} - \frac{1}{M_*}\left\{ -M_* + M_* \ln(M_{TOT} - M_*) + M_{TOT} \ln\left(\frac{M_{TOT}}{M_{TOT} - M_*}\right)\right\}.$$

$$= 1 + \ln M_{TOT} - \ln(M_{TOT} - M_*)$$

$$- \frac{M_{TOT}}{M_*} \ln\left(\frac{M_{TOT}}{M_{TOT} - M_*}\right)$$

$$= 1 + \ln\left(\frac{M_{TOT}}{M_{TOT} - M_*}\right) \cdot \left(1 - \frac{M_{TOT}}{M_*}\right)$$

Note $\mu = \dfrac{M_{TOT} - M_*}{M_{TOT}} = 1 - \dfrac{M_*}{M_{TOT}}$

$1 - \mu = \dfrac{M_*}{M_{TOT}}$ (star fraction)

$$\frac{M_{TOT}}{M_*} = \frac{1}{1-\mu}$$

$$1 - \frac{M_{TOT}}{M_*} = 1 - \frac{1}{1-\mu} = \frac{1-\mu-1}{1-\mu}$$

$\Rightarrow$

$$\frac{\langle Z_*\rangle}{p} = 1 + \left(\ln \frac{1}{\mu}\right) \cdot \frac{-\mu}{1-\mu} \qquad (\ln\tfrac{1}{\mu} = -\ln\mu)$$

$$\Rightarrow \frac{\langle Z_*\rangle}{p} = 1 + \frac{\mu}{1-\mu} \ln\mu$$

$$\longrightarrow 1 \quad \text{as} \quad \mu \to 0 \qquad \checkmark$$

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2. Problem 4.10. Allow for inflow of metal free gas. No longer closed box! Set inflow rate proportional to star-formation rate.

$$\Delta M_* + \Delta M_g = \nu \Delta M_*.$$

Equation 4.14 needs a modification:

$$\Delta Z = \Delta\left(\frac{M_h}{M_g}\right) = \frac{\Delta M_h}{M_g} - \frac{M_h}{M_g^2}\Delta M_g$$

as before

$$= p\frac{\Delta M_*}{M_g} - \frac{Z\Delta M_*}{M_g} - Z\frac{\Delta M_g}{M_g}$$

But now we have

$$\Delta M_g = (\nu - 1)\,\Delta M_*$$

<u>not</u> $\Delta M_g = -\Delta M_*$

hence

$$\Delta Z = p\frac{\Delta M_*}{M_g} - Z\frac{\Delta M_*}{M_g} - Z(\nu-1)\frac{\Delta M_*}{M_g}$$

$$= p\frac{\Delta M_*}{M_g} - Z\frac{\Delta M_*}{M_g}(1+\nu-1)$$

$$= (p - \nu Z)\frac{\Delta M_*}{M_g} \quad \checkmark$$

$$= \frac{(p - \nu Z)}{(\nu - 1)}\frac{\Delta M_g}{M_g} \quad \checkmark$$

$$\Rightarrow \quad dZ = \frac{p - \nu Z}{\nu - 1}\frac{dM_g}{M_g}$$

Substitute $y = p - \nu Z$ $\quad dy = -\nu\, dZ$

$$dZ = -\frac{1}{\nu}\,dy$$

We then have the equation

$$-\frac{1}{\nu} dy = \frac{y}{\nu-1} \frac{dM_g}{M_g}$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{\nu}{1-\nu} \frac{dM_g}{M_g}$$

$$\Rightarrow \quad d\ln y = \frac{\nu}{1-\nu} \cdot d\ln M_g$$

Integrate:

$$\ln y/y(t=0) = \frac{\nu}{1-\nu} \ln \frac{M_g}{M_g(t=0)}$$

$$\Rightarrow \quad \frac{y}{y(t=0)} = \left(\frac{M_g}{M_g(t=0)}\right)^{\nu/1-\nu}$$

$$y = p - \nu Z \qquad y(t=0) = p \qquad (Z(t=0) = 0)$$

$$\Rightarrow \quad \frac{p-\nu Z}{p} = \left(\frac{M_g}{M_g(t=0)}\right)^{\nu/1-\nu}$$

$$1 - \frac{\nu}{p} Z = \left(M_g/M_g(t=0)\right)^{\nu/1-\nu}$$

$$\Rightarrow \quad Z(t) = \frac{P}{2} \left\{ 1 - \left(\frac{M_g(t)}{M_g(t=0)} \right)^{2/1-\nu} \right\}$$

- 4.21 ✓

$$Z(t) \leq \frac{P}{2}.$$

The "effective yield" $P \to \frac{P}{2}$.