Invariant characterization of a wave.

$$I_1 = F_{\mu\nu} F^{\mu\nu} \propto \underline{E}^2 - \underline{B}^2$$

$$I_2 = F_{\mu\nu} {}^*F^{\mu\nu} \propto \underline{E} \cdot \underline{B}$$

For a freely propagating wave:

$$I_1 = I_2 = 0$$

Relativistic mechanics

Newtons free mass point

$$\frac{d\underline{p}}{dt} = \underline{F} \quad \left\{ \begin{array}{l} \underline{p} = m\underline{u} \\ E = E(0) + \frac{1}{2} m \underline{u}^2 \end{array} \right. \quad (\underline{u}^2 \ll c^2)$$

Ignored in Newtonian approx. Work-energy thm involves change in E.

Find a Lorentz covariant form of the eqs. of motion. Only 3 independent dynamical eqs.: $E = \underline{P}^2/2m$ is expressible through $\underline{P}$.

Embed $\underline{p}, E$ into a four vector?

$$p_\mu = \left(\frac{E}{c} , \underline{p} \right)$$

Measure time in the rest frame of the particle: <u>proper time</u>, τ.

The proper time is <u>invariant</u>:

$$c\,d\tau = \sqrt{c^2 dt^2 - d\underline{x}^2}$$

(In the rest frame: $d\underline{x} = 0$)

Four vector form for <u>free</u> motion:

$$\frac{d p_\mu}{d\tau} = 0$$

Four velocity $p_\mu = m u_\mu$

Rest mass

Constraint:

$$\frac{d}{d\tau} p_\mu p^\mu = 0$$

Use dimensional argument: mc is the only quantity of dim. (momentum)

$$p_\mu p^\mu = (mc)^2$$

Numerical factor??
(Use Newtonian limit)

~~Conjecture~~ Conjecture that $p_\mu p^\mu = (mc)^2$ holds in the presence of forces too

Energy and momentum

$$p_0 = \frac{E}{c}$$

$$p^2 = \frac{E^2}{c^2} - \underline{p}^2 = (mc)^2$$

~~Gets~~ $E = c\sqrt{(mc)^2 + \underline{p}^2}$

Newtonian limit:

$$E = \sqrt{m^2c^4 + c^2\underline{p}^2}$$

$$= mc^2 \sqrt{1 + \frac{\underline{p}^2}{m^2c^2}}$$

$$\approx mc^2 \left(1 + \frac{1}{2}\frac{\underline{p}^2}{m^2c^2} + \cdots\right)$$

$$= mc^2 + \frac{\underline{p}^2}{2m} + \cdots$$

Now conjecture

$$\frac{dp_\mu}{d\tau} = F_\mu , \quad p^2 = (mc)^2$$

$$F_\mu = (\, ? \, , \, \underline{F} \,);$$

however $\quad p^2 \equiv p_\mu p^\mu = \text{const}$

requires $\quad p_\mu F^\mu = 0$

$$p_0 F_0 - \underline{p} \cdot \underline{F} = 0$$

$$F_0 = \frac{\underline{p} \cdot \underline{F}}{p_0} = c \, \frac{\underline{p}}{E} \cdot \underline{F}$$

$$\cancel{\underline{p}/E = \underline{v} \Rightarrow F_0 = c \ldots \text{power}}$$

~~Now~~ $$\frac{\underline{p}}{\sqrt{m^2c^4 + c^2\underline{p}^2}} = \frac{1}{c} \, \frac{\underline{p}}{\sqrt{m^2c^2 + \underline{p}^2}} = \frac{\underline{v}}{c} ,$$

thus $\quad F_0 = \underline{v} \cdot \underline{F} \quad$ (power)

Thus, $$\frac{d\underline{p}}{d\tau} = \underline{F}$$

$$\frac{1}{c} \frac{dE}{d\tau} = F_0 = \underline{v} \cdot \underline{F}$$

(Work-energy thm)

~~Stdusr~~ ~~Ex.~~ Independent components using system time.

$$p_\mu = m\frac{dx^\mu}{d\tau}$$

$$d\tau = \sqrt{dt^2 - dx^2/c^2}$$

$$= dt\sqrt{1-\beta^2}$$

$$\underline{P} = \frac{m\underline{v}}{\sqrt{1-v^2/c^2}}$$

$$\frac{d\underline{P}}{dt} = \sqrt{1-\underline{v}^2/c^2}\,\underline{F}$$

$$\boxed{\frac{d}{dt}\frac{m\underline{v}}{\sqrt{1-v^2/c^2}} = \underline{F}\sqrt{1-\underline{v}^2/c^2}}$$

(Recognize that

$$\frac{|\underline{v}|/c}{\sqrt{1-v^2/c^2}} = \sinh y,$$

$$\frac{1}{\sqrt{1-v^2/c^2}} = \cosh y,$$

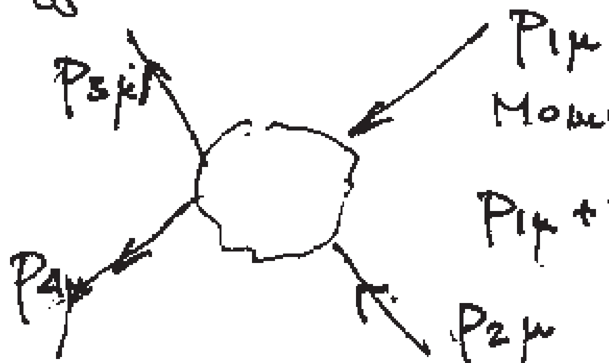
which may be useful for 1-d. motion)

Final: 12/2/ 9-12
Rm. 259

Collisions (elastic)

Short range, central forces ⇒ planar motion. (Conservation of angular momentum.)
Consequence: outside of the range of forces, the motion is free.

An elastic collision means that both energy and momentum are conserved, i.e.

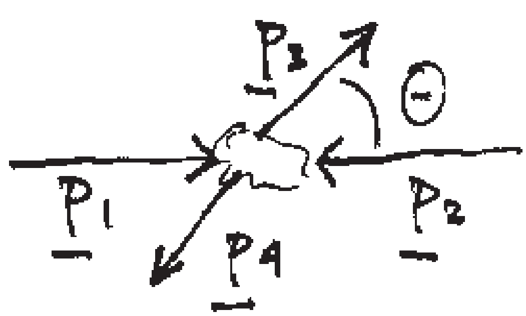


Momentum conservation:

$$p_{1\mu} + p_{2\mu} = p_{3\mu} + p_{4\mu}$$

Go to the "center of momentum system" (CMS):

$$\underline{p}_1 + \underline{p}_2 = \underline{p}_3 + \underline{p}_4 = 0$$



Number of variables needed to describe the collision:

$p_{1\mu} \cdots\cdots p_{4\mu}$	16	variables
$p_1^2 = m_1^2c^2 \cdots p_4^2 = m_4^2c^2$	-4	constraints
$p_{1\mu} + p_{2\mu} = p_{3\mu} + p_{4\mu}$	-4	constraints
Choice of Lorentz frame	-3	(β)
Orientation of coord. sys	-3	(3 angles)
Total:	2	free variables

Typical choice: i) scattering angle Θ in CMS ii) Energy in CMS, E

Simplify the algebra: take $m_1 = \cdots = m_4 = m$

Lorentz invariants:

$$\left.\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \end{aligned}\right\}$$ (S. Mandelstam, 195

The first invariant, s.

Physical meaning?

In the CMS:

$$s = (p_{10} + p_{20})^2 - (\underline{p}_1 + \underline{p}_2)^2$$

$$\underline{p}_1 + \underline{p}_2 = 0 \text{ in CMS}$$

$$p_{10} + p_{20} = \frac{1}{c}(E_1 + E_2) \leftarrow \text{Energy in CMS}$$

Thus in the CMS:

$$s = \frac{1}{c^2}\left(2\sqrt{c^2\underline{p}^2 + m^2c^4}\right)^2$$

$$c^2 s = 4\,(c^2\underline{p}^2 + m^2c^4)$$

Hence, the magnitude of the momentum in the CMS:

$$s = 4\,(\underline{p}^2 + m^2c^2)$$

$$\boxed{\underline{p}^2 = \frac{s}{4} - m^2c^2}$$

Energy of one particle in CMS:

$$E = \sqrt{c^2\underline{p}^2 + m^2c^4}$$

$$\boxed{E = \frac{1}{2}\sqrt{c^2 s}} = \frac{1}{2}c\sqrt{s}$$

Express in terms of quantities in the laboratory system (LS)

$$p_{10L} = mc \qquad \underline{p}_{1L} = 0$$

$$p_{2L} = \left(\sqrt{\underline{p}_L^2 + m^2c^2}\,,\ \underline{p}_L\right)$$

$$s = (p_1 + p_2)^2 = \left(mc + \sqrt{\underline{p}_L^2 + m^2c^2}\right)^2 - \underline{p}_L^2$$

$$s = m^2c^2 + 2mc\underbrace{\sqrt{\underline{p}_L^2 + m^2c^2}}_{= E_L/c} + m^2c^2$$

$$s = 2\left(m^2c^2 + m E_L\right)$$

Note that if $E_L \gg mc^2$, then

$$s \sim 2 m E_L$$

<u>The second invariant, t.</u>

Again, go to the CMS.

$$t = (p_3 - p_1)^2 = \underbrace{(p_{30} - p_{10})^2}_{=0,\ \text{because I chose } m_3 = m_1} - (\underline{p}_3 - \underline{p}_1)^2$$

However: $|\underline{p}_3| = |\underline{p}_1| = |\underline{p}|$ thus,

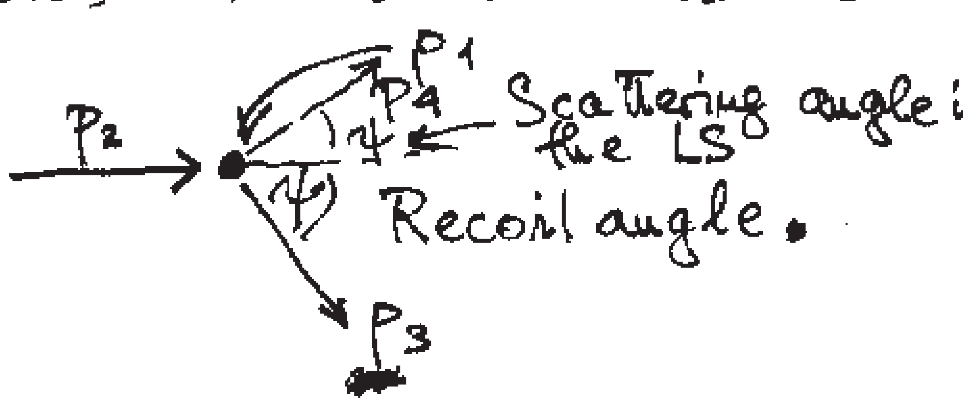
$$t = -(2\underline{p}^2 - 2\,\underline{p}_3 \cdot \underline{p}_1)$$

$$\underline{p}_3 \cdot \underline{p}_1 = \underline{p}^2 \cos\Theta,$$

$$t = -2\underline{p}^2(1 - \cos\Theta) = -4\underline{p}^2 \sin^2\frac{\Theta}{2}$$

⇒ [Apart from the (−) sign, which is a matter of convention, $\sqrt{-t}$ is just the momentum transfer in the CMS] ⇐

As before, write down t in the LS.



$$t = (p_{1L} - p_{3L})^2 = \left(mc - \frac{1}{c}E_3\right)^2 - \vec{p}_{3L}^2$$

$$= m^2c^2 + \frac{1}{c^2}\left(c^2\vec{p}_3^{\,2} + m^2c^4\right) - 2mE_3 - \vec{p}_3^{\,2}$$

$$t = 2m^2c^2 - 2mE_3$$

$$= 2m\left(mc^2 - E_3\right)$$

This expresses t in terms of the recoil energy, which is a rather useful form.
Exploiting energy and momentum conservation, other forms are possible too.

Dynamics of charged particles

(Jackson, Ch 12)

$$\frac{du^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} u_\beta$$

$$p^\alpha = m u^\alpha$$

"Natural units" : $c = 1$

The reasons for a Lagrangian formulation. (First, a particle in an ext. field, next: particle + field)

i) Variational principles lead to Euler eqs whose compatibility is guaranteed.

ii) A Lagrangian formulation allows the discovery of conservation laws.

iii) The action plays a central role in relat. quantum theory.

iv) Possible approx. technique.

$\Rightarrow$ Review of Hamilton's principle, non-relativistic mechanics.

Conservative system:

$$L = T - V$$

$$S = \int_{t_1}^{t_2} dt \, L(q, \dot{q})$$

$$\delta S = \int dt \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) \qquad \left(\delta \dot{q} = \frac{d\,\delta q}{dt} \right)$$

Boundary value problem: determine the trajectory with given $q(t_1) = q_1$ and $q(t_2) = q_2$.

$$\Rightarrow \delta q(t_1) = \delta q(t_2) = 0$$

Therefore, integrate by parts:

$$\frac{\delta L}{\delta \dot{q}} \delta \dot{q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \delta q \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}$$

$$\frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2} = 0$$

$$\delta S = \int_{t_1}^{t_2} dt \, \delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right)$$

$\delta S = 0$ gives the Euler-Lagrange eqs. since δq is arbitrary:

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

Non-relativistic free particle:

$$L = \frac{1}{2} m \frac{dx^a}{dt} \frac{dx^b}{dt} \delta_{ab}$$

Relativistic generalization?

<u>Guiding principles</u>:

- Lorentz covariance
  $\Rightarrow$ Cannot use system time to parametrize trajectory
- Parametrization is irrelevant
  $\Rightarrow$ reparametrization invariance.
- Free particle obeys a linear equation of motion, $\underline{p} = \text{const}$ (Newton's 1st law)
  $\Rightarrow$ quadratic Lagrangian.

Try for free particle:

$$S = \frac{1}{2} \int ds \sqrt{e} \left(\frac{1}{e} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \eta_{\mu\nu} + m^2 \right)$$

Constant, Lorentz invariant "potentia

The variable "e" serves to assure reparametrization invariance:

New parametrization:

$$s = f(s')$$

$$ds = f'(s')\,ds'$$

$$\sqrt{e}\,ds(\cdots) = \sqrt{e'}\,ds'(\cdots)$$

$$= f'(s')\sqrt{e'}\,ds'(\cdots)$$

$$e = (f'(s'))^2\, e'$$

or

$$e' = \frac{e}{\left(\frac{df}{ds'}\right)^2}$$

Extremize w.r. to e

$$\delta S = \frac{1}{2}\int ds \left(-\frac{1}{2} e^{-3/2} u^2 + m^2 \frac{1}{2} e^{-1/2} \right) \delta e$$

$$u^2 \equiv \eta_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

$$\delta S = \frac{1}{4}\int ds \sqrt{e}\, \delta e \left(-\frac{1}{e^2} u^2 + m^2 \frac{1}{e} \right) = 0$$

The quantity e is not a dynamical variable: $\frac{de}{ds}$ does not occur.

Eliminate e:

$$e = \frac{u^2}{m^2}$$

Substitute in:

$$S = -\frac{1}{2}\int ds \left(\frac{u^2}{m^2}\right)^{1/2} \left[\frac{m^2}{u^2} u^2 + m^2\right]$$

$$= -m \int ds \left(\eta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}\right)^{1/2}$$

in conventional units: mc

(the conventional formula, for $m \neq 0$). However, this formalism is OK for $m = 0$ as well.

(Exercise: set up the Lagrangian formalism for $m = 0$.)

Variation w.r. to x^α:

$$m \frac{d}{ds}\left[\frac{dx^\alpha/ds}{\left(\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds}\eta_{\rho\sigma}\right)^{1/2}}\right] = 0$$

If I choose $s = \tau$, then

$$\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\eta_{\rho\sigma} = 1 \quad (= c^2 \text{ in conv. units})$$

then $\frac{d}{d\tau} p^\alpha = 0$ (Newton's 1st law)

Lorentz force: $F^\alpha = F^{\alpha\beta} u_\beta$

Guess:

$$S = -\int ds \sqrt{e} \left[\frac{1}{2} \frac{1}{e} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \eta_{\mu\nu} + \frac{1}{2} m^2 + q \frac{dx^\alpha}{ds} \frac{1}{\sqrt{e}} A_\alpha \right]$$

Eliminate e as before,

$$S = -\int ds \left(m \left(\frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \eta_{\alpha\beta} \right)^{1/2} + q \frac{dx^\alpha}{ds} A_\alpha (x(s)) \right)$$

Variation of the interaction term

$$\delta \int ds \frac{dx^\alpha}{ds} A_\alpha (x(s))$$

$$= \int ds \left(\frac{d}{ds} \delta x^\alpha A_\alpha + \frac{dx^\alpha}{ds} \frac{\partial A_\alpha}{\partial x^\mu} \delta x^\mu \right)$$

Integrate by parts, drop the integrated part:

$$\delta \int ds \frac{dx^\alpha}{ds} A_\alpha = \int ds \left(-\frac{d}{ds} A_\alpha \delta x^\alpha + \frac{dx^\alpha}{ds} \frac{\partial A_\alpha}{\partial x^\mu} \delta x^\mu \right) = ($$

$$(*) = \int ds\, \delta x^\mu \left(-\frac{\partial A_\mu}{\partial x^\nu} \frac{dx^\nu}{ds} + \frac{dx^\nu}{ds} \frac{\partial A_\nu}{\partial x^\mu} \right)$$

$$= \int ds\, \delta x^\mu \frac{dx^\nu}{ds} F_{\nu\mu} ,$$

so that the correct eq. of motion is obtained.

Verify gauge invariance:

$$\left(\underline{A}' = \underline{A} + \underline{\nabla} \Lambda , \quad \phi' = \phi - \frac{\partial \Lambda}{\partial t} \right)$$

$$\Rightarrow \quad A'_\mu = A_\mu + \partial_\mu \Lambda$$

$$\int \frac{dx^\alpha}{ds} \left(A_\alpha + \partial_\alpha \Lambda \right) ds$$

$$= \int ds \frac{\partial x^\alpha}{ds} A_\alpha + \int ds \frac{dx^\alpha}{ds} \partial_\alpha \Lambda$$

However:

$$\int ds \frac{dx^\alpha}{ds} \partial_\alpha \Lambda = \int_{s_1}^{s_2} ds \frac{d\Lambda}{ds} = \Lambda(x(s_2)) - \Lambda(x(s_1))$$

Class of gauge functions:

$$\Lambda(x(s_2)) = \Lambda(x(s_1)) = 0$$

Using system time:

$$S = \cancel{q}\int dt \left[-mc\left(1 - \underline{u}^2/c^2\right)^{1/2} + \frac{q}{c}\left(A_0 + \underline{u}\cdot\underline{A}\right)\right] \qquad \left(\underline{u} = \frac{d\underline{x}}{dt}\right)$$

$$= \cancel{q}\int dt \left[-mc\left(1 - \underline{u}^2/c^2\right)^{1/2} + \frac{q}{c}\,\underline{u}\cdot\underline{A} - q\phi\right]$$

Static, homogeneous magnetic field, $\underline{B}$.

$$A_a = -\frac{1}{2}\epsilon_{ars} x_r B_s$$

$(\partial_a A_a = 0$, Coulomb gauge)

Eqs. of motion:

$$\frac{d\underline{p}}{dt} = \frac{e}{c}\,\underline{v}\times\underline{B}$$

$$\frac{dE}{dt} = 0.$$

Non-covariant (re-) formulation, charged particle in a homogeneous magnetic field.

$$S = \int (-d\tau) \left[mc \left(\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \eta_{\mu\nu} \right)^{1/2} \right.$$

$$\left. + \frac{q}{c} \frac{dx^\alpha}{d\tau} A_\alpha \right]$$

Conventional choice

Choose system time as a variable:

$$d\tau = \left(dt^2 - \frac{1}{c^2} d\underline{x}^2 \right)^{1/2}$$

$$= dt \left(1 - u^2/c^2 \right)^{1/2}$$

$$\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \eta_{\mu\nu}$$

$$= \frac{1}{1 - u^2/c^2} \left(\left(\frac{dt}{dt} \right)^2 - \left(\frac{d\underline{x}}{dt} \right)^2 \right) = 1$$

$$S = - \int dt \left[mc \left(1 - u^2/c^2 \right)^{1/2} \right.$$

$$\left. + \frac{q}{c} \frac{dx^\alpha}{dt} A_\alpha \right]$$

$$\frac{dE}{dt} = \frac{d}{dt}\frac{mc^2}{\sqrt{1-\underline{v}^2/c^2}} = 0$$

$$\Rightarrow |\underline{v}| = \text{const}$$

$$\underline{P} = \frac{m\underline{v}}{\sqrt{1-\underline{v}^2/c^2}}$$

$$\frac{d\underline{v}}{dt} = \underline{v} \times \underline{\omega}_B$$

$$\underline{\omega}_B = \frac{e\underline{B}}{\gamma m c} = \frac{ec\underline{B}}{E}$$

Decompose $\underline{v}$ into a component ∥ and perpendicular to $\underline{\omega}$.

$$\frac{dv_\parallel}{dt} = 0 ; \quad (\text{Choose } v_\parallel = 0)$$

Choose ω to be along the z axis;

$$\frac{dv_1}{dt} = v_2 \omega_B , \quad \frac{dv_2}{dt} = -v_1 \omega_B$$

$v_\perp^2 = v_+ v_-$ $\quad v_\pm = v_1 \pm i v_2 , \quad \frac{dv_\pm}{dt} = \mp i \omega_B v_\pm$

$$v_\pm = v_0 e^{\mp i\omega_B t}$$

v_0 is the magnitude of $v_\perp$

$$\frac{dx_\pm}{dt} = v_0 \, e^{\mp i \omega_B t}$$

$$x_\pm = \pm \frac{i v_0}{\omega_B} e^{\mp i \omega_B t}$$

Express v_0 : $E_\perp = \frac{mc^2}{\sqrt{1 - v_0^2/c^2}}$, to get

$$x_\pm = \pm i \frac{c p_\perp}{e|\underline{B}|} e^{\mp i \omega t}$$

The gyration radius is:

$$\boxed{R_L = \frac{c p_\perp}{e|\underline{B}|}}$$

(Larmor radius)