171.201 Final Exam Solutions

Fall 2003

NOTE: The mean score was 117 / 200

① (a) Pulse travels at group velocity $V_g = \frac{d\omega}{dk} = \left(\frac{P}{\rho_s}\right)^{1/2}$

Total distance travelled $= 2D$

$$\Rightarrow 2D = t\left(\frac{P}{\rho_s}\right)^{1/2} \Rightarrow \boxed{t = 2D\left(\frac{\rho_s}{P}\right)^{1/2}}$$

(b) Yes, the pulse will have the same shape.

Since the phase velocity $V_p = \frac{\omega}{k} = \left(\frac{P}{\rho_s}\right)^{1/2}$

is the same as the group velocity, there is no dispersion of the pulse. If it were not the case that $\omega \propto k$, then the two velocities would be different & the pulse would spread.

(c) Reflection coefficient,

$$R_{so} = \frac{Z_s - Z_o}{Z_s + Z_o} = \frac{\sqrt{\rho_s} - \sqrt{\rho_o}}{\sqrt{\rho_s} + \sqrt{\rho_o}}$$

$$\text{Amplitude} = AR = \boxed{A\left(\frac{\sqrt{\rho_s} - \sqrt{\rho_o}}{\sqrt{\rho_s} + \sqrt{\rho_o}}\right)}$$

(d) The two waves will interfere destructively if their path lengths differ by $(n+\frac{1}{2})\lambda$.

Reflection coefficient from bottom of oil is:

$$R_{OR} = \left(\frac{\sqrt{\rho_0} - \sqrt{\rho_R}}{\sqrt{\rho_0} + \sqrt{\rho_R}} \right)$$

(Both R's are negative.)

$$\Rightarrow 2L = (n + \tfrac{1}{2}) \lambda$$

$$2L = (n + \tfrac{1}{2}) \frac{2\pi}{K_0}$$

$$\cancel{L = \frac{\pi}{\omega}\sqrt{\rho_0}}$$

$$\boxed{L = \frac{\pi}{\omega} \sqrt{\frac{P}{\rho_0}} \left(n + \frac{1}{2}\right)}$$

(e) For the destructive ~~wave~~ interference to give $\emptyset$, we need the reflected waves to have same amplitude

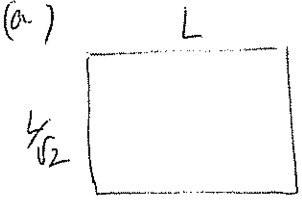
$$R_{SO} = T_{SO} R_{OR} T_{OS}$$

if $R << 1$, then $T \approx 1 \Rightarrow R_{SO} = R_{OR}$

$$\frac{\sqrt{\rho_S} - \sqrt{\rho_0}}{\sqrt{\rho_S} + \sqrt{\rho_0}} = \frac{\sqrt{\rho_0} - \sqrt{\rho_R}}{\sqrt{\rho_0} + \sqrt{\rho_R}}$$

or, $\rho_0 = \sqrt{\rho_S \rho_R}$

2. (a)



This is the generalisation of a 1-D wave eqn. problem.

In 2-D the wave eqn. becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

for which the solution ~~is~~:

$$\psi(x,y,t) = e^{i[k_x x + k_y y \mp \omega t]}$$

For fixed edges $\psi(x,0,0) = 0$, $\psi(x, L/\sqrt{2}, 0) = 0$
$\psi(0,y,0) = 0$ $\psi(L, y, 0) = 0$

The normal modes are

$$\psi(x,y,t) = \sin\frac{n\pi}{L}x \sin\frac{m\pi}{L/\sqrt{2}}y \cos\omega t$$

where $k_x = \frac{n\pi}{L}$, $k_y = \frac{m\pi}{L/\sqrt{2}}$, $n, m = 1, 2, 3 \ldots$

and

$$\omega^2 = v_p^2\left(\frac{n^2\pi^2}{L^2} + \frac{2m^2\pi^2}{L^2}\right)$$

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$$= \frac{v_p^2\pi^2}{L^2}(n^2 + 2m^2)$$

$$\omega = \frac{v_p\pi}{L}\sqrt{n^2+2m^2}$$

n	m	$\sqrt{n^2+2m^2}$
1	0	1
0	1	$\sqrt{2}$
1	1	$\sqrt{3}$
2	0	2
2	1	$\sqrt{6}$

(b)

If one of the sides of length L is held fixed,

$$\left.\frac{\partial \psi}{\partial x}\right|_{x=0,L} = 0 \qquad \text{while} \quad \psi(x, 0, t) = 0$$
$$\psi(x, L/\sqrt{2}, t) = 0 \quad \text{remains true}$$

(5) $k_y = \frac{m\pi}{4\sqrt{L}}$ remains.

normal modes now are

$$\psi(x, y, t) = [\cos k_x x \sin k_y y] \cos \omega t$$

$$k_x L = 0 \Rightarrow k_x L = \frac{n\pi}{2}$$

$$k_x = \frac{n\pi}{2L}, \quad n = 1, 2, \ldots$$

(5) $$\Rightarrow \omega = \frac{v_p \pi}{L}\left(\sqrt{\frac{n^2}{4} + 2m^2}\right)$$

n	m	$\sqrt{\frac{n^2}{4}+2m^2}$
0	1	$\sqrt{2}$
1	0	$1/2$

3 (a) Energy & Momentum are conserved in the decay:

$$E_i = E_f \qquad P_i = 0 = P_f = P_\mu - P_\nu$$

$$\Downarrow \qquad \text{but, } E_\nu = P_\nu c = P_\mu c$$

$$M_{0x} c^2 = E_\mu + E_\nu$$

$$\Rightarrow M_{0x} c^2 = \frac{M_{0\mu}}{\sqrt{1-v^2/c^2}} c^2 + P_\nu c = \frac{M_{0\mu}}{\sqrt{1-v^2/c^2}} \left(c^2 + vc \right)$$

$$M_{0x} = 4 M_{0\mu} \left(1 + \frac{\sqrt{15}}{4} \right) = \boxed{400 \left(1 + \frac{\sqrt{15}}{4} \right) \text{MeV}}$$

(b) $$P_\nu = P_\mu = \frac{M_{0\mu}}{\sqrt{1-v^2/c^2}} vc = 4 M_{0\mu} \frac{\sqrt{15}}{4} c = \boxed{\sqrt{15}\ 100\ \frac{\text{MeV}}{c}}$$

(c) The muon lasts 2×10^{-6} sec in its rest frame. This time is dilated in the earth's frame:

$$\tau_{earth} = \frac{2\times10^{-6}\ \text{sec}}{\sqrt{1-v^2/c^2}} = 8\times10^{-6}\ \text{sec}$$

The time it takes the muon to travel 10 km is

$$10^4\ m = \frac{\sqrt{15}}{4} (3\times10^8\ m/s)\ t$$

$$t \simeq 3\times10^{-5}\ \text{sec.}$$

$$\text{Probability} = e^{-t/\tau} = e^{-3\times10^{-5}/8\times10^{-6}} \simeq \boxed{e^{-4}}$$

4/a.

$$M\ell \frac{d^2\theta_1}{dt^2} = -Mg\sin\theta_1 + k\ell(\theta_2 - \theta_1)$$

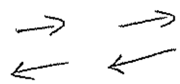
$$M\ell \frac{d^2\theta_2}{dt^2} = -Mg\sin\theta_2 - k\ell(\theta_2 - \theta_1)$$

small amplitude approximation: $\sin\theta \sim \theta$

$$\frac{d^2\theta_1}{dt^2} = -\frac{g}{\ell}\theta_1 + \frac{k}{M}(\theta_2 - \theta_1)$$

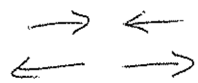
$$\frac{d^2\theta_2}{dt^2} = -\frac{g}{\ell}\theta_2 - \frac{k}{M}(\theta_2 - \theta_1)$$

b mode 1: spring essentially doesn't function and the masses move together $(\theta_1 = \theta_2)$



$$\omega_1^2 = \frac{g}{\ell}$$

mode 2: $\theta_1 = -\theta_2$



$$\omega_2^2 = \frac{g}{\ell} + \frac{2k}{M}$$

c. when $\theta_2 = \theta_0$, $\theta_1 = 0$

this occurs at $\frac{1}{4}$ beat period

where beat period is $\frac{2\pi}{\omega_{beat}} = \frac{2\pi}{\omega_1 - \omega_2}$

so, $t = \frac{1}{4}\left(\frac{2\pi}{\omega_1 - \omega_2}\right)$

5a. similar to Young's double slit, but phase difference comes before arriving at slits, not as a consequence of the slits

$$I(\theta) = I_o \cos^2 \frac{1}{2}\left(\frac{2\pi d \sin\theta}{\lambda}\right)$$

so, in this case

$$I = I_o \cos^2\left(\frac{\pi d \sin\alpha_A}{\lambda}\right)$$

b. the phase difference is given by $\Delta\phi = \frac{2\pi \sin\alpha}{\lambda}$

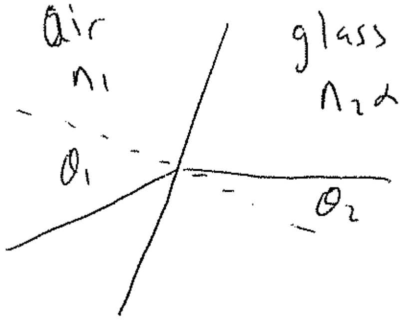
- want a phase difference of π for a minima to occur

$$\frac{2\pi\, d \sin\alpha_A}{\lambda} = \pi + \frac{2\pi d \sin\alpha_B}{\lambda}$$

solving for d

$$d = \lambda/2(\sin\alpha_A - \sin\alpha_B)$$

(6) (a) Snell's law states $\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1}$



θ_1 is the same for all the colors. $\Rightarrow \sin\theta_2 \sim \frac{1}{n_2} \sim \frac{1}{\lambda^2}$

$\Rightarrow \theta_2$ is smallest for red, largest for blue.

$\Rightarrow$ red is bent most; blue least

$\Rightarrow$ $\boxed{\text{red} = C, \text{ green} = B, \text{ blue} = A}$

(b) The first minimum in the diffraction pattern occurs at $\lambda = s\sin(\theta)$ $\Rightarrow$ A has bigger λ than B

$\Rightarrow$ $\boxed{A \text{ red}, B \text{ blue}}$

(c) $I_{trans} = I_0 \cos^2\theta$

$\Rightarrow$ To minimize transmission we want to maximize the angle ~~of~~ between the easy axes of successive ~~pto~~ polarizers. $\Rightarrow$ It's best to have $\boxed{BAC \text{ or } CAB}$

To maximize transmission we want to minimize the angle between successive easy axes. ~~Fo~~

$\Rightarrow$ It's best to have $\boxed{ACB \text{ or } BCA}$

(d) Happy Holidays!