171.201 Fall 2004

Final Exam Solutions

(1) (a) $y(x)$ is an antisymmetric function. Hence, it should be comprised only of sine terms $\Rightarrow$ $\boxed{\dfrac{A_n}{B_n} = 0}$

(b) $B_n = \frac{2}{L}\int_{-L/2}^{L/2} y(x)\sin(k_n x)\,dx$

NOTE: boundary conditions mean that allowed values of k are: $k_n = \frac{2\pi}{L} n$

$$= \frac{2}{L}\int_{-L/4}^{0}(-h)\sin(k_n x)\,dx + \frac{2}{L}\int_{0}^{L/4} h\sin(k_n x)\,dx$$

$$B_n = -\frac{2}{L}\left(2h\,\frac{1}{k_n}\cos(k_n x)\Big|_0^{L/4}\right) = \frac{4h}{\cancel{L}}\frac{\cancel{L}}{2\pi n}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{2h}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)$$

$\Rightarrow$ $B_1 = \frac{2h}{\pi}$, $B_2 = \frac{2h}{\pi}$, $B_3 = \frac{2h}{3\pi}$, $B_4 = 0$ (ie mode not excited)

$B_5 = \frac{2h}{5\pi}$ with $\lambda_n = \frac{L}{n}$

(1) (c) Yes, each normal mode oscillates at frequency

$$\omega_n = c k_n = \sqrt{\frac{T}{\rho}} \frac{2\pi}{L} n$$

since all frequencies are integer multiples of fundamental $(n=1)$, all modes will be at their $t=0$ position whenever the fundamental is. This happens once each period:

$$T_1 = \frac{2\pi}{\omega_1} = 2\pi \sqrt{\frac{\rho}{T}} \frac{L}{2\pi} = \boxed{L \sqrt{\frac{\rho}{T}}}$$

(2) (a) $\langle S \rangle = \frac{c}{8\pi} E_0^2 \hat{z}$

$$\bar{S} = \frac{c}{4\pi} \frac{1}{\mu} \bar{E} \times \bar{B} = \frac{c}{4\pi} \bar{E}^2 \hat{z} \quad \text{in vacuum}$$

circularly polarized wave:

$$\bar{E} = E_{max} \left\{ \cos(\omega t - kz) \hat{y} + \cos(\omega t - kz + \tfrac{\pi}{2}) \right\} \hat{x}$$

$$\bar{E}^2 = E_{max}^2 \left\{ \cos^2(\omega t - kz) + \cos^2(\omega t - kz + \tfrac{\pi}{2}) \right\}$$

$$\langle \bar{E}^2 \rangle = E_{max}^2 \left(\tfrac{1}{2} + \tfrac{1}{2} \right)$$

$$\Rightarrow \langle S \rangle = \frac{c}{4\pi} E_{max}^2 \Longrightarrow E_{max} = \frac{E_0}{\sqrt{2}}$$

so, $\boxed{\bar{E} = \frac{E_0}{\sqrt{2}} \left\{ \cos(\omega t - kz) \hat{y} + \cos(\omega t - kz + \tfrac{\pi}{2}) \hat{x} \right\}}$

(b) Polarizer with easy axis along x-direction transmits only the component of the $\bar{E}$ field that is parallel to $\hat{x}$:

$$\boxed{\bar{E} = \frac{E_0}{\sqrt{2}} \cos(\omega t - kz + \tfrac{\pi}{2}) \hat{x}}$$

(c) Polarizer along $\hat{y}$ would only transmit $\bar{E}$ field parallel to $\hat{y}$

But $\bar{E} \parallel \hat{x} \Rightarrow \boxed{I = 0}$

(d) $$E_{final} = \frac{E_0}{\sqrt{2}} \cos\theta \sin\theta \cos(\omega t - kz + \tfrac{\pi}{2}) \hat{y}$$

$$\Rightarrow \langle S \rangle_{final} = \frac{c}{4\pi} \frac{E_0^2}{2} \cos^2\theta \sin^2\theta \, \frac{1}{2} = \boxed{\frac{c}{16\pi} E_0^2 \cos^2\theta \sin^2\theta}$$

(3) (a) There are two events

	Alice	Ben
firecracker A	$x_A = 0,\ t_A = 0$	$x'_A = 0,\ t'_A = 0$
firecracker B	$x_B = 50\,m,\ t_B = 10^{-7}\,s$	$x'_B = ??\ ,\ t'_B = 0$

if $t' = 0$ for event B, then:

$$t' = \frac{t - \left(\frac{v}{c^2}\right)x}{\sqrt{1 - v^2/c^2}}$$

$$0 = \frac{(10^{-7}\,s) - \frac{v}{c^2}(50 \text{ meters})}{\sqrt{1 - v^2/c^2}}$$

$$\frac{v}{c^2}\,50 = 10^{-7}$$

$$\frac{v}{c} = \frac{10^{-7} \times 3 \times 10^{8}}{50} = \frac{3}{5}$$

$$\boxed{v = \frac{3}{5}c}$$

(b) Length contraction:

$$x'_B - x'_A = \sqrt{1 - v^2/c^2}\,(x_B - x_A) = \sqrt{1 - \frac{3^2}{5^2}}\ 50$$

$$= \boxed{40 \text{ meters}}$$

(c) $$t = \frac{40 \text{ meters}}{c} = \boxed{\frac{4}{3} \times 10^{-7} \text{ sec}}$$

4 (a) If initial intensity is I_0, then $I_0/4$ will reach the detector via mirror 1 and $I_0/4$ will reach via mirror 2. The two waves from mirror 1 and mirror 2 will interfere. Their path length difference is $2(x_1 - x_2)$

$$\Rightarrow \text{phase difference} = \phi = \frac{2\pi}{\lambda} 2(x_1 - x_2) = \frac{4\pi}{\lambda}(x_1 - x_2)$$

Interference intensity is:

$$I = \left(I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2}\cos\phi\right) \qquad I_1 = I_2 = I_0/4$$

$$= \frac{I_0}{2}(1 + \cos\phi) = \frac{I_0}{2}\left(1 + \cos\left(\frac{4\pi}{\lambda}(x_1 - x_2)\right)\right)$$

(b) The idea here is that the index of refraction of the air inside the chamber, leads to a change in wavelength and therefore to a change in phase of the wave from mirror 2 that interferes with that from mirror 1.

$$\omega = vk \qquad v = \frac{c}{n} \Rightarrow k = n\underbrace{k_0}_{\text{wave vector in vacuum}} = \frac{2\pi n}{\lambda_0}$$

The data shows 3 periods, i.e. ϕ changes by 6π during evacuation, or the number of wavelengths inside the chamber changes by 3:

$$(\text{final \# of wavelengths}) - (\text{initial \#}) = \frac{\overbrace{2d}^{\text{wave traversed chamber twice}}}{\lambda_{final}} - \frac{2d}{\lambda_{initial}} = 3$$

$$\lambda_{initial} \equiv \lambda = \frac{n}{\lambda_0}$$

$$\lambda_{final} = \lambda_0 = \frac{n}{\lambda}$$

$$\frac{2dn}{\lambda} - \frac{2d}{\lambda} = 3$$

$$n - 1 = \frac{3\lambda}{2d} \Rightarrow \boxed{n = \frac{3\lambda}{2d} + 1}$$

Final Exam Problem #5 Solution

5) *Note the similarity of this problem to Assignment #6, problem 4 (Pain 4.12)*
a) The voltage along the bottom wire is constant; we are free to set it to zero. Then by the relation charge = capacitance·voltage, we have

$$q_1 = CV_1 \ \ , \ q_2 = CV_2 \ \ , \ q_3 = CV_3. \tag{1}$$

From the diagram it is clear that the currents are directly related to changes in these charges:

$$\dot{q}_1 = -I_a \ \ , \ \dot{q}_2 = I_a - I_b \ \ , \dot{q}_3 = I_b \tag{2}$$

From the inductor law, voltage = inductance (rate of change of current), we have

$$L\dot{I}_a = V_1 - V_2 \ \ , \ L\dot{I}_b = V_2 - V_3. \tag{3}$$

Combining Eqs. (3) with Eqs. (1), and working in terms of the charges on the capacitors, we get

$$-L\ddot{q}_1 = \frac{q_1}{C} - \frac{q_2}{C} \ \ , \ L\ddot{q}_3 = \frac{q_2}{C} - \frac{q_3}{C}. \tag{4}$$

Since we want to work with currents, differentiate Equations (4) once more to obtain

$$\begin{aligned} -L\left(\frac{\partial^3 q_1}{\partial t^3}\right) &= L\ddot{I}_a = \frac{-I_a}{C} - \frac{I_a - I_b}{C}, \\ L\left(\frac{\partial^3 q_3}{\partial t^3}\right) &= L\ddot{I}_b = \frac{I_a - I_b}{C} - \frac{I_b}{C}. \end{aligned} \tag{5}$$

Cleaning up, we have for our solution

$$\ddot{I}_a = -2\frac{I_a}{LC} + \frac{I_b}{LC} \ \ , \ \ddot{I}_b = \frac{I_a}{LC} - 2\frac{I_b}{LC} \tag{6}$$

b) Easiest using the matrix methods from class. Several people forgot the minus sign in front of the 2×2 matrix, this leads to errors.

$$\begin{bmatrix} \ddot{I}_a \\ \ddot{I}_b \end{bmatrix} = -\begin{bmatrix} 2/LC & -1/LC \\ -1/LC & 2/LC \end{bmatrix} \begin{bmatrix} I_a \\ I_b \end{bmatrix} = -\bar{R}\begin{bmatrix} I_a \\ I_b \end{bmatrix}. \tag{7}$$

The frequencies of the normal modes are obtainable by solving the *eigenvalue equation* $det(\bar{R} - \bar{I}\omega^2) = 0$:

$$\begin{vmatrix} 2/LC - \omega^2 & -1/LC \\ -1/LC & 2/LC - \omega^2 \end{vmatrix} = (2/LC - \omega^2) - (1/LC)^2 = 0. \tag{8}$$

The solutions by inspection are $\omega_1^2 = 1/LC$, $\omega_2^2 = 3/LC$, so the normal mode frequencies are

$$\omega_1 = \sqrt{\frac{1}{LC}} \ \ , \ \omega_2 = \sqrt{\frac{3}{LC}}. \tag{9}$$

(Alternate, equivalent solution) Guess the normal variables $Y_1 = I_a + I_b$, $Y_2 = I_a - I_b$. The coupled Eqs. (6) then become

$$\begin{aligned} \ddot{Y}_1 &= -\frac{I_a}{LC} - \frac{I_b}{LC} \ \ , & \ddot{Y}_2 &= -3\frac{I_a}{LC} - 3\frac{I_b}{LC} \\ \ddot{Y}_1 &= -\frac{Y_1}{LC} \ \ , & \ddot{Y}_2 &= -\frac{3Y_2}{LC} \end{aligned} \tag{10}$$

The normal mode freqencies are clearly the same.

c) The shapes of the the normal modes $I_a(t)/I_b(t)$ can be obtained by solving the *eigenvector equations*

$$(\bar{R} - \bar{I}\omega_1^2)\begin{bmatrix} I_a \\ I_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad , \quad (\bar{R} - \bar{I}\omega_2^2)\begin{bmatrix} I_a \\ I_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

for I_a/I_b. For the first mode ($\omega = \omega_1$), this becomes

$$\begin{bmatrix} 1/LC & -1/LC \\ -1/LC & 1/LC \end{bmatrix}\begin{bmatrix} I_a \\ I_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{12}$$

which gives $\boldsymbol{I_a = I_b}$. For the second mode ($\omega = \omega_2$), we have

$$\begin{bmatrix} -1/LC & -1/LC \\ -1/LC & -1/LC \end{bmatrix}\begin{bmatrix} I_a \\ I_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{13}$$

which gives $\boldsymbol{I_a = -I_b}$. These shapes could also be guessed from the symmetry of the circuit.

d) The *most* general solution for the currents in this circuit is

$$\begin{aligned} I_a(t) &= A cos(\omega_1 t) + B sin(\omega_1 t) + C cos(\omega_2 t) + D sin(\omega_2 t), \\ I_b(t) &= A cos(\omega_1 t) + B sin(\omega_1 t) - C cos(\omega_2 t) - D sin(\omega_2 t) \end{aligned} \tag{14}$$

This form avoids the need for "phase angles" - they are absorbed into the relative values of A/B and C/D. The exact values of A, B, C, D are set by the initial conditions. Handling the derivatives of the currents first, we have

$$\begin{aligned} \left(\frac{dI_a}{dt}\right)_{(t=0)} &= 0 \Longrightarrow B\omega_1 + D\omega_2 = 0. \\ \left(\frac{dI_b}{dt}\right)_{(t=0)} &= 0 \Longrightarrow B\omega_1 - D\omega_2 = 0. \end{aligned} \tag{15}$$

These 2 conditions require $B = D = 0$ So the allowed currents are restricted to

$$\begin{aligned} I_a(t) &= A cos(\omega_1 t) + C cos(\omega_2 t), \\ I_b(t) &= A cos(\omega_1 t) - C cos(\omega_2 t) \end{aligned} \tag{16}$$

The initial currents ($I_a(0) = I_0$,$I_b(0) = 0$) require $A + C = I_0$, $A - C = 0$. The solution is $A = I_0/2$, $C = I_0/2$. So equal currents flow in both modes and

$$I_a(t) = \frac{I_0}{2}\left[cos\left(t\sqrt{\frac{1}{LC}}\right) + cos\left(t\sqrt{\frac{3}{LC}}\right)\right] \quad , \quad I_b(t) = \frac{I_0}{2}\left[cos\left(t\sqrt{\frac{1}{LC}}\right) - cos\left(t\sqrt{\frac{3}{LC}}\right)\right] \tag{17}$$

You were asked **"At what times will $\boldsymbol{I_b(t) = I_0}$?"**
$I_b(t) = I_0$ ***if and only if*** the quantity in the right hand set of square brackets is equal to 2. This can occur *if and only if*

$$cos\left(t\sqrt{\frac{1}{LC}}\right) = 1 \text{ and } cos\left(t\sqrt{\frac{3}{LC}}\right) = -1. \tag{18}$$

This requires

$$t\sqrt{1/LC} = 2m\pi \text{ and } t\sqrt{3/LC} = (2n+1)\pi, \tag{19}$$

where m and n are integers. Dividing yields the condition

$$\sqrt{3} = \frac{2n+1}{2m}, \tag{20}$$

which has no solution, since $\sqrt{3}$ is irrational. **Therefore $\boldsymbol{I_b(t) = I_0}$ never happens.**

Many students used one of of the two following arguments, which seem logical but are in fact spurious:

i) "Beats": The line of argument follows Pain p. 85-89:

$$I_a(t) = I_0 cos\left(\frac{\omega_2 - \omega_1}{2}t\right) cos\left(\frac{\omega_2 + \omega_1}{2}t\right) \tag{21}$$

The beat frequency $\omega_b = (\omega_2 + \omega_1)/2$ defines an "envelope" for the oscillation of $I_a(t)$. The beat period is then $T = 2\pi/\omega_B$. Then we have an exact relation for times when I_a must be zero:

$$I_a = 0 \text{ for } t = 0, T, 2T, \ldots \tag{22}$$

Relation 22 is indeed true. **Unfortunately, many students made the faulty conclusion that $I_b = I_0$ at these times.** This is incorrect because the conditions for $I_b = I_0$ and $I_a = 0$ are different. The first condition is given by equations (18), which as shown have no solution. However, $I_a = 0$ whenever

$$\begin{aligned} cos\left(t\sqrt{\frac{1}{LC}}\right) + cos\left(t\sqrt{\frac{3}{LC}}\right) &= 0 \\ t\sqrt{\frac{1}{LC}} + (2g+1)\pi &= t\sqrt{\frac{3}{LC}} \\ t &= \frac{(2g+1)\pi}{(\sqrt{3}-1)\sqrt{LC}} \end{aligned} \tag{23}$$

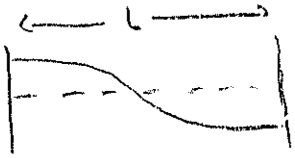
for any integer g. The subtlety here is that $I_b = I_0 \Longrightarrow I_a = 0$, but $I_a = 0 \not\Rightarrow I_b = I_0$.

ii) Energy Conservation: Students assumed that *"when $I_a = 0$, $I_b = I_0$ because all the energy has been transferred from one oscillation to the other"*. The conceptual error here may be too closely identifying this circuit with the coupled-pendulum system shown on Pain p. 85-89. In any case, it is best to regard energy as being stored in the normal-mode currents $Y_1 = I_a + I_b$ and $Y_2 = I_a - I_b$ rather than the separate loop currents (I_a, I_b). Energy may *not* be transferred between the two normal modes. The form of the energy is

$$E = aY_1^2 + b\dot{Y}_1^2 + cY_2^2 + d\dot{Y}_2^2, \tag{24}$$

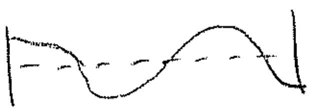
so the energy of the system is a sum over terms proportional to $(I_a \pm I_b)^2$ or $(\dot{I}_a \pm \dot{I}_b)^2$. Clearly caution is required here. One could calculate the times when $I_b = I_0$ from $E(t)$ and energy conservation, but as shown above no such times exist.

(6) (a) The boundary conditions for sound waves in a tube with open end require that waves have an anti-node at the end:



$\lambda = 2L$

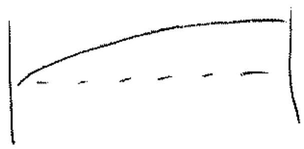
$\lambda = L$



$\lambda = \frac{2}{3}L$

in general, $\boxed{\lambda = \frac{2L}{n}}$

(b) For fixed end, boundary condition requires a node:



$\lambda = 4L$

$\lambda = \frac{4}{3}L$

$\lambda = \frac{4}{5}L$

in general, $\boxed{\lambda = \frac{4L}{2n-1}}$

(c) In the home expt you found the length L at which the tube resonated due to standing wave. Resonance dominated by fundamental. Therefore, $\lambda = 2L$

call tuning fork frequency: ν_{Fork}

standing wave: $A \sin(kx)\cos(\omega t)$ where $\omega = vK$ (speed of sound)

but we have $K = \frac{2\pi}{\lambda} = \frac{2\pi}{2L} = \frac{\pi}{L}$ $\Rightarrow$ $v = \frac{2\pi \nu_{Fork}}{\pi/L} = \boxed{\frac{2\nu_{Fork}}{L}}$

$\omega = 2\pi \nu_{Fork}$

⑦

(a) $\boxed{A}$ This curve is the only one that maintains a |slope| > 1 at all times so that $v < c$ at all times.

(b) $\boxed{4}$ $I \propto \dfrac{\sin^2(N\phi/2)}{\sin^2(\phi/2)}$ Since there are 3 zeros between each peak, the numerator must go to zero 4 times for every time the denominator does. Hence, $N = 4$.

(c) $\boxed{C}$ The velocity of the pulse is given by the group velocity $v_g = \frac{d\omega}{dk}$. Since $\omega(k)$ has downward curvature $\frac{d\omega}{dk}$ decreases with increasing k. Since $\frac{d\omega}{dk} \neq$ constant the pulse disperses.

(d) $\boxed{C}$ Since the electron & positron both travel with $v < c$, one can boost to a frame where the net momentum of the two is 0. However, this can not be done for the photon since $v = c$. Hence, momentum is not conserved.