Physics 171.201
Final Exam

December 19th, 2005

Answer all **seven** problems. Be sure that you pace yourself so that you have enough time to work on each problem. Note that the problems do not have equal weight. Partial credit will be given, so be sure to **show your work** as clearly as possible. Good luck!

List of potentially useful formulae

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$E = mc^2$$

$$z' = z$$

$$\vec{p} = m\vec{v}$$

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}$$

$$E^2 = m_0{}^2c^4 + p^2c^2$$

$$u_x{}' = \frac{u_x - v}{1 - u_x v/c^2}$$

$$E^2 - p^2c^2 = E'^2 - p'^2c^2$$

$$u_y{}' = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2}$$

Maxwell's Equations

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = \frac{-1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

Poynting Vector: $\vec{S} = \frac{c}{4\pi}\frac{1}{\mu}\vec{E} \times \vec{B}$

EM Energy Density: $U = \frac{1}{8\pi}\left(\varepsilon\vec{E}^2 + \frac{1}{\mu}\vec{B}^2\right)$

Snell's Law: $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

Potentially useful formulae, continued:

General diff. eq. for oscillators: $$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

has solutions

$$x(t) = A e^{-\gamma t/2} \cos\left(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\, t + \alpha\right) + x_{ss}(t) \qquad \omega_0 > \frac{\gamma}{2}$$

$$= (A + Bt) e^{-\gamma t/2} + x_{ss}(t) \qquad \omega_0 = \frac{\gamma}{2}$$

$$= A e^{-\Gamma_1 t} + B e^{-\Gamma_2 t} + x_{ss}(t) \qquad \omega_0 < \frac{\gamma}{2}$$

where $$\Gamma_1 = \frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}, \quad \Gamma_2 = \frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

and the steady state solution is

$$x_{ss}(t) = A(\omega) \cos(\omega t - \delta(\omega))$$

where $$A(\omega) = \frac{f_0}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2\right]^{1/2}} \qquad \tan \delta(\omega) = \frac{\gamma \omega}{\omega_0^2 - \omega^2}$$

The wave equation $$\frac{\partial^2}{\partial x^2} y(x,t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} y(x,t)$$

where, eg, $v = \sqrt{T/\rho}$ for a string

Two-dimensional wave eq: $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2}$$

Reflection & Transmission of EM waves

For $\bar{E}$ field: $$R = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad T = \frac{2 Z_2}{Z_1 + Z_2}$$

For $\bar{B}$ field: $$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad T = \frac{2 Z_1}{Z_1 + Z_2}$$

where $Z = \frac{1}{n}$ (n = index of refraction)

Problem 1 (20 points)

Consider a room in the shape of a perfect three-dimensional cube of side L. The air in the room can support sound waves, which can be thought of as waves of pressure. The velocity of sound in the room is v. The walls of the room are rigid, which creates the boundary condition that the first derivative of the pressure at the walls must be zero. Mathematically, we can express this boundary condition as:

$$\frac{\partial P}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = L$$
$$\frac{\partial P}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = L$$
$$\frac{\partial P}{\partial z} = 0 \text{ at } z = 0 \text{ and } z = L$$

(a) Solve the three-dimensional wave equation to derive an expression for the standing sound waves in the room.

(b) In terms of L and v, determine the values of the four lowest frequencies that standing waves in the room can have. Specify which frequencies are degenerate.

(c) The ambient pressure in the room is P_0. For each of the four standing wave frequencies in part (b), one observes that the maximum pressure anywhere in the room is $P_0+\Delta$. For each frequency write down the maximum pressure at the center of the room.

Problem 2 (40 points)

A wave pulse is moving down a string to the right as in the figure (a) shown below. The pulse has a trapezoidal shape with height h and three segments each of length d as shown. The string is under tension T and has mass density ρ.

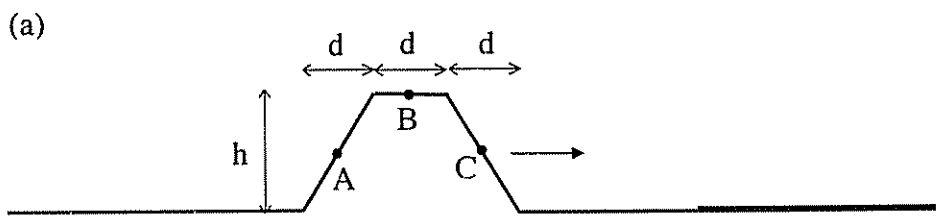
(a) Consider small pieces of the string located at the points A, B, and C shown in the picture. Give the magnitude and direction of the velocity of each piece.

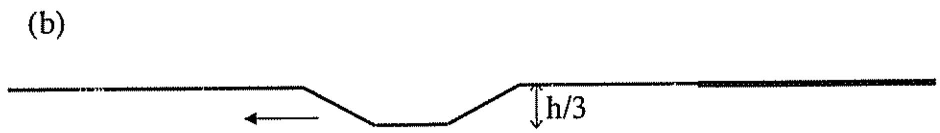
(b) The pulse is incident on a second string that is attached to the first string. The second string is also under tension T but has a different mass density. Figure (b) below shows a snapshot of the pulse reflected from the interface between the two strings. In terms of ρ, what is the density of the second string?

(c) What is the shape and amplitude of the pulse that is transmitted down the second string?

(d) Considering again the pulse in figure (a). Determine the kinetic and potential energy of the pulse.

(e) Imagine that the pulse in figure (a) was generated by somebody pulling upward and then downward on the end of the string. Determine the vertical force as a function of time that the person exerted to generate the pulse.





Problem 3 (30 points)

Two antennae, placed a distance δ from one another as shown below, emit electromagnetic waves with wavelength λ. The antennae emit the waves in phase with one another. A detector is placed a distance r from the midpoint between the antennae as shown below.

(a) Determine the difference in path length from the two antennae to the detector as a function of the angle θ shown in the figure. Assume $r \gg \delta$.

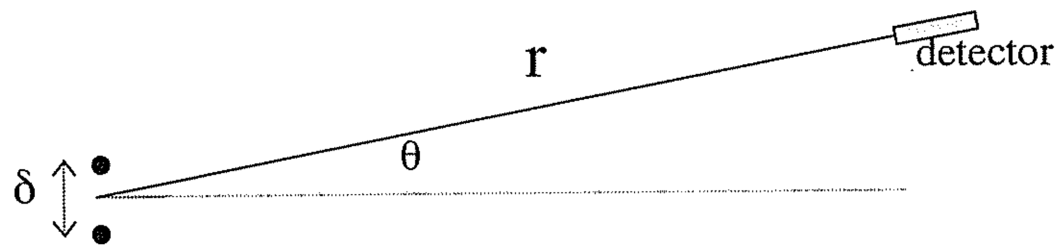
(b) Based on your result for part (a) show that the intensity measured by the detector as a function θ is:

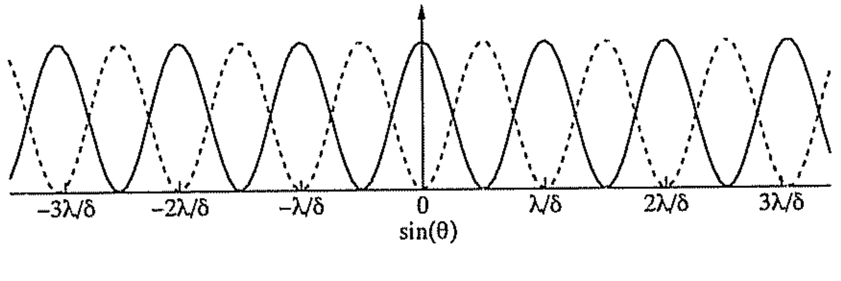
$$I(\theta) = I_{max} \cos^2\left(\frac{\pi\delta}{\lambda}\sin\theta\right)$$

This result is shown in the graph below by the solid curve. [Hint: Recall the trigonometric identity $\cos(2x) = 2\cos^2(x) - 1$.]

(c) Imagine instead that the antennae emit their waves precisely out of phase with one another. Give the intensity as a function of θ for this case.

(d) Again consider the case in which the antennae emit in phase. Imagine that a slab of nonabsorbing material of thickness T is placed in the front of one of the antennae. The intensity measured by the detector changes from the solid curve in the graph to the dashed curve. Note we are assuming $\lambda \ll \delta$ so that θ is very small. What is the index of refraction of the material?





Problem 4 (30 points)

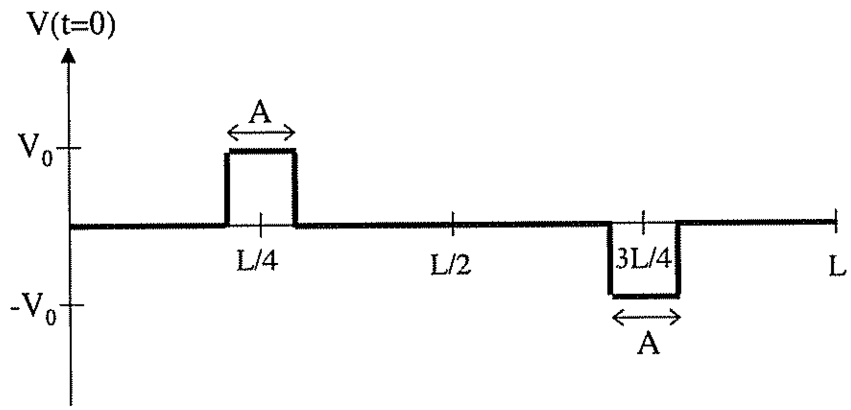
A flexible string of length L is stretched with an equilibrium tension T between fixed supports. Its mass per unit length is ρ. The string is set into vibrations by two hammers that simultaneously strike blows at time $t = 0$ impulsively imparting a transverse velocity V_0 to small segments of the string of length A. One hammer strike is centered at a point L/4 from the left end and imparts an upward velocity, and the other hammer strike is centered at a point 3L/4 from the left end and imparts a downward velocity. Below is a graph showing the resulting velocity of the string as a function of position at $t = 0$. Recall that the vibrations of the string can be considered a superposition of normal modes with wavelengths $\lambda = 2L/n$ where n=1, 2, 3, etc.

(a) What is the wavelength of the lowest normal mode (that is, the longest wavelength mode) that is NOT excited by the hammer?

(b) What is the amplitude of the lowest normal mode with nonzero amplitude?

(c) Until the hammers strike at $t = 0$, none of the string is displaced from its equilibrium position. At what time will the string return to such a configuration?

(d) Call the time you found in part (c) t_{min}. Draw a sketch of the shape of the string at the instant $t_{min}/2$. [Note: this question is a bit tricky, but the solution does not require any elaborate calculations.]



Problem 5 (30 points)

Imagine a high-energy physics experiment to search for exotic particles. In the experiment an electron moves at the very high velocity $\frac{4}{5}c$ and collides into a stationary electron. The idea is for the electrons to annihilate, creating a new particle. Call the rest mass of the electron M_0.

(a) In terms of M_0, what is total energy in the laboratory frame before the collision?

(b) What is the total energy in the center of mass frame before the collision? [Hint: use Lorentz invariants. Do not use relativistic velocity addition.]

(c) What is the rest mass of the particle created by this collision?

(d) Imagine instead that the experiment is arranged so that the electrons have equal and opposite velocities in the laboratory. If the energy in the lab frame is the same as you calculated in part (a), what is the magnitude of the velocities?

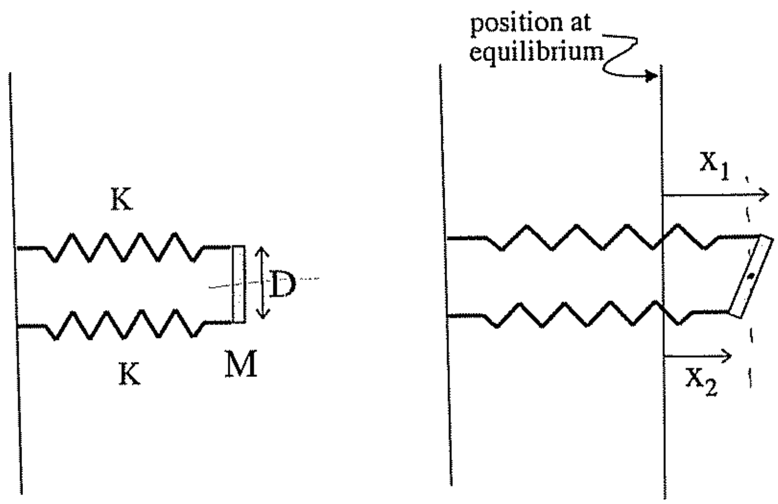
(e) For this second arrangement, what is the rest mass of the particle created by this collision? Currently, there is a proposal to build a high-energy physics experiment called the International Linear Collider that will use this arrangement. Can you give a one-sentence explanation as to why this arrangement is better than the first one?

Problem 6 (20 points)

Consider two identical springs with spring constant K. One end of each spring is attached to the wall. The ends of the springs are attached to opposite end of a rigid rod of mass M and length D as shown in the picture on the left below. The rod has moment of inertia I. The springs can expand and contract but cannot rotate. These springs can be considered a pair of coupled oscillators. The objective of this problem is to find the normal mode frequencies.

(a) Call the extension of the upper spring x_1 and the extension of the lower spring x_2 as shown in the figure on the right. Write down the equation of motion for the center of mass of the rod. Write down a second equation describing the motion about the center of mass. (For the second equation, consider the torque on the rod and use $I\ddot{\theta} = \sum \text{torques}$.) You should be able to write both equations in terms of x_1 and x_2.

(b) Using the results of part (a), find the normal mode frequencies.



Probem 7 (30 points)

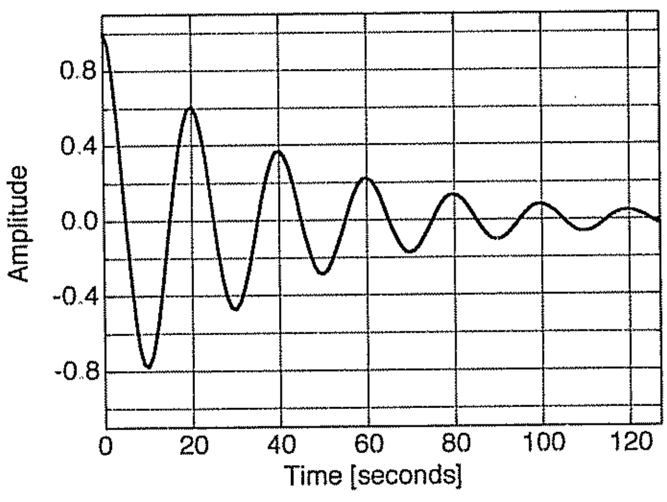
In each of the following four short problems, be sure to explain your answer in order to earn full credit.

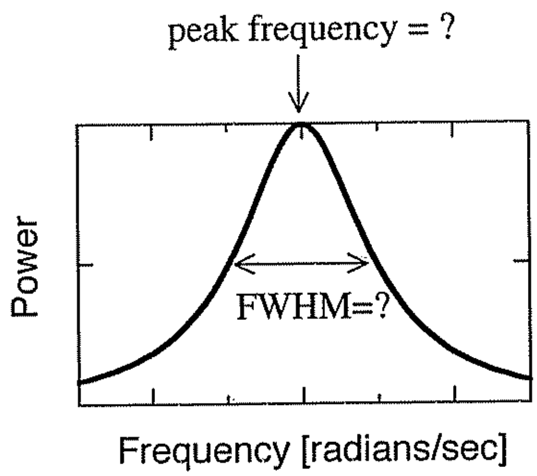
(a) (8 points) Consider waves traveling in a dispersive medium with dispersion relation

$$\omega = Ak + Bk^3$$

Both A and B are positive constants. Which is larger, the group velocity or the phase velocity?

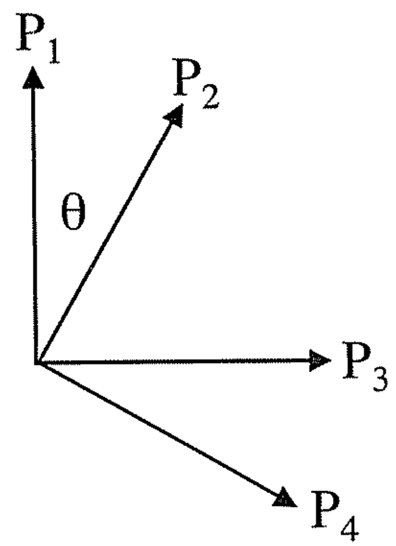
(b) (8 points) Consider a damped harmonic oscillator. A measurement is made of the amplitude as a function of time. The data is shown in the graph on the left. When an oscillatory force is applied to the oscillator it absorbs power at a rate that depends on the frequency of the force as shown in the graph on the right. Based on the data in the left-hand graph, estimate values for the peak frequency and full width at half maximum of the power resonance.





Problem 7 (continued)

(c) **(8 points)** A circularly polarized beam of light with intensity I_0 is incident on a stack of four ideal polarizers. The light goes through the polarizers labeled P1, P2, P3, P4 in that order. Polarizers P1 and P3 are crossed at 90°. Polarizers P2 and P4 are also crossed at 90° but are rotated by an angle θ with respect to P1 and P3. What is the intensity of the light that is transmitted through the stack of polarizers? How would you reorder the stack so that the transmitted intensity is zero?



(d) **(6 points)** In your opinion were the labs a worthwhile experience? If given a chance, what you change about them? Do you think the course should have more or fewer labs, keeping in mind that no section is held in the weeks when there is a lab? (Hint: There is no wrong answer to this question.)