

Physics 171.201 Final Exam

December 19th, 2006

Answer all **six** problems. Be sure that you pace yourself so that you have enough time to work on each problem. Note that the problems do not have equal weight. Partial credit will be given, so be sure to **show your work** as clearly as possible. Good luck!

List of potentially useful formulae

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$E = mc^2$$

$$z' = z$$

$$\vec{p} = m\vec{v}$$

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}$$

$$E^2 = m_0^2 c^4 + p^2 c^2$$

$$u_x' = \frac{u_x - v}{1 - u_x v/c^2}$$

$$E^2 - p^2 c^2 = E'^2 - p'^2 c^2$$

$$u_y' = \frac{u_y \sqrt{1 - v^2/c^2}}{1 - u_x v/c^2}$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \times \vec{E} = \frac{-1}{c} \frac{\partial \vec{B}}{\partial t}$$

Maxwell's Equations

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

Poynting Vector: $\vec{S} = \frac{c}{4\pi} \frac{1}{\mu} \vec{E} \times \vec{B}$

EM Energy Density: $U = \frac{1}{8\pi} \left(\epsilon \vec{E}^2 + \frac{1}{\mu} \vec{B}^2 \right)$

Snell's Law: $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

Potentially useful formulae, continued:

General differential equation for force, damped harmonic oscillator:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F_0 \cos(\omega t)$$

has solutions,

$$x(t) = \begin{cases} Ae^{-\gamma t/2} \cos\left(\left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2} t + \alpha\right) + x_{ss}(t) & \omega_0 > \gamma/2 \\ (A + Bt)e^{-\gamma t/2} + x_{ss}(t) & \omega_0 = \gamma/2 \\ Ae^{-\Gamma_1 t} + Be^{-\Gamma_2 t} + x_{ss}(t) & \omega_0 < \gamma/2 \end{cases}$$

where $\Gamma_1 = \frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$ and $\Gamma_2 = \frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$

and the steady state solution is: $x_{ss}(t) = A(\omega) \cos(\omega t - \delta(\omega))$

where ,

$$A(\omega) = \frac{F_0}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2\right]^{1/2}} \quad \text{and} \quad \tan(\delta(\omega)) = \frac{\gamma \omega}{\omega_0^2 - \omega^2}$$

The wave equation: $\frac{\partial^2}{\partial x^2} y(x,t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} y(x,t)$

where, e. g., $v = \sqrt{T/\rho}$ a string.

Two dimensional wave equation: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 x}{\partial t^2}$

Reflection and transmission of EM waves:

For E field: $R = \frac{Z_2 - Z_1}{Z_2 + Z_1}$ $T = \frac{2Z_2}{Z_1 + Z_2}$

For B field: $R = \frac{Z_1 - Z_2}{Z_2 + Z_1}$ $T = \frac{2Z_1}{Z_1 + Z_2}$

where $Z=1/n$ (n = index of refraction)

Problem 1 (35 points)

Consider a string of length L with linear mass density ρ and under tension T . The string is fixed at the two ends $x = 0$ and $x = L$. The center of the string is pulled to a small displacement h and held at rest as shown in the picture below. At $t = 0$ the string is released. The initial conditions for the string can thus be written as:

$$y(x,0) = \begin{cases} 2hx/L & x < L/2 \\ 2h(1-x/L) & x > L/2 \end{cases}$$
$$\dot{y}(x,0) = 0$$

The displacement of the string at any time following its release can be written as a sum of normal modes:

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L) \cos(\omega_n t + \phi_n)$$

- Which Fourier coefficients B_n are equal to zero? (You should not need to perform any integrals to answer this question.)
- For the terms with non-zero coefficients, all of the phases ϕ_n have the same value. What is it? (Again, you should not need to perform any integrals.)
- Calculate explicitly the values of the Fourier coefficients and show that for the non-zero B_n , $|B_n| = 8h/(n\pi)^2$. You may find the following integral useful:
 $\int x \sin(x) dx = -x \cos(x) + \sin(x)$.
- Find the total vibrational energy in the string by determining the potential energy stored in the string at $t = 0$.
- One can show that the energy in the n^{th} normal mode of the string is given by:

$$E_n = \frac{1}{4} \rho L \omega_n^2 B_n^2$$

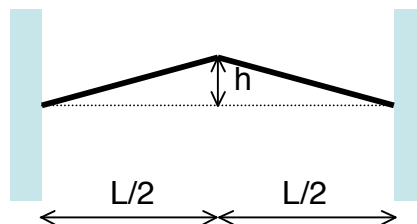
Determine the total vibrational energy of the string using this expression. One of the following relations may be helpful:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}$$



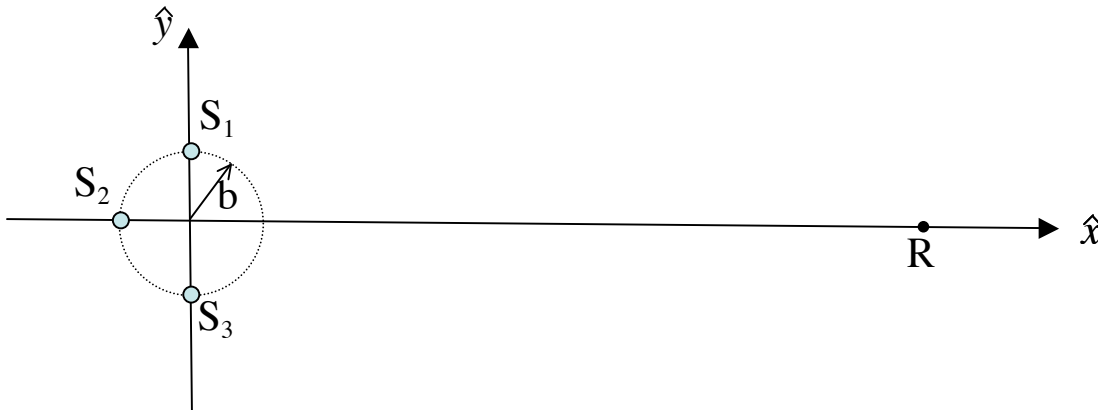
Problem 2 (25 points)

Three identical point sources of electromagnetic waves are situated on the circumference of a circle of radius b centered at the origin as shown in the top picture below. The sources oscillate in phase with one another emitting electromagnetic waves that have wavelength λ and that are polarized along the z -direction. If only one of the sources were emitting waves, the time-averaged intensity in the x - y plane a distance R ($R \gg \lambda$ and $R \gg b$) from the origin would be I_0 .

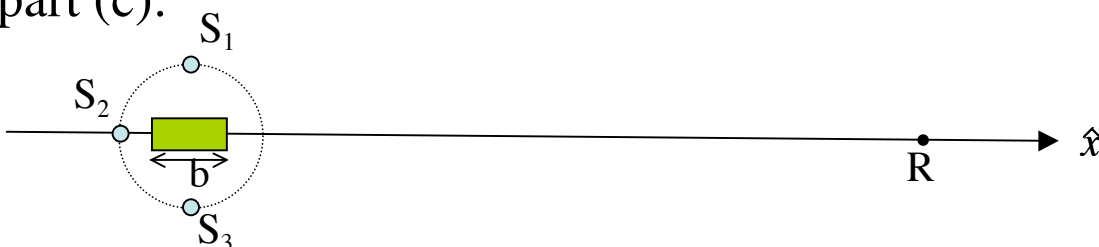
- (a) Calculate the time-averaged intensity that would be measured at a distance R from the origin along the x -axis when all three sources are emitting, assuming $R \gg \lambda$ and $R \gg b$. Express your result in terms of I_0 , λ , and b . Depending on how you approach this problem, you may find the following trig identity helpful:

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

- (b) Sketch a plot of the intensity as a function of b for fixed λ . Identify clearly the values and positions of the maxima and minima.
- (c) Imagine now that the radius b is fixed at $b = \lambda/2$, and a piece of dielectric material with index of refraction n and length b is placed along the path of the waves traveling from S_2 to R , as shown in the bottom picture below. Sketch a plot of the intensity as a function of n . Again, identify clearly the values and positions of the maxima and minima.



part (c):



Problem 3 (30 points)

The picture below on the left shows a simple pendulum comprised of a light rigid rod of length ℓ supporting a mass M .

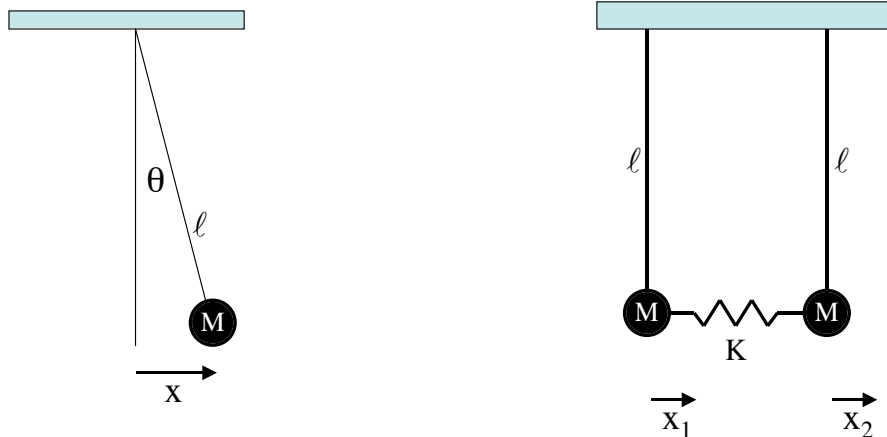
- (a) Show that if the angular displacement θ of the pendulum is small, then the equation of motion for the mass can be written in terms of the lateral displacement x as: $\ddot{x} + \frac{g}{\ell}x = 0$.

Consider now two identical such pendulums coupled together by a spring of stiffness K as shown below on the right. Call the displacement of the mass on the left x_1 and that of the mass on the right x_2 .

- (b) Write down the coupled differential equations for the motion of each of the masses that describe small amplitude displacements from their equilibrium positions.
- (c) Determine the normal mode frequencies for the system.
- (d) Imagine that the two masses are initially at rest in equilibrium. At time $t = 0$, a bullet strikes the mass on the left imparting a sudden impulse that gives the mass a velocity v_0 . The displacements and velocities of the masses at the instant after the bullet strikes are then:

$$\begin{aligned} x_1(0) &= 0 & x_2(0) &= 0 \\ \dot{x}_1(0) &= v_0 & \dot{x}_2(0) &= 0 \end{aligned}$$

Find the ratio between the amplitudes of the two normal modes that describes the subsequent motion of the masses.



Problem 4 (30 points)

As part of its search for extrasolar planets, NASA discovers a planet that appears to be very much like earth orbiting a star 40 light years from our solar system. An expedition is planned to send astronauts to the planet. NASA would like the astronauts to age no more than 30 years during the journey. In this problem, neglect any issues related to the acceleration of the astronauts' spaceship.

NOTE: A light year is a unit of distance. It is how far light travels in one year. For example, a spaceship with a velocity of $0.5c$ will require 80 years to travel 40 light years.

- (a) At what velocity must the astronaut's spaceship travel in earth's reference frame so that the astronauts age 30 years during the journey? [Hint: Measured in years, the trip according to an observer on earth will take a time $\Delta t = 40c/v$.
- (b) According to the astronauts in the spaceship, what will be the distance of their journey?
- (c) In order to let people on earth keep track of their progress, the astronauts shine a light pulse back to earth once each year, as measured by the clocks on the spaceship. How often do the light pulses reach earth?
- (d) In order to keep in touch with the astronauts, the people on earth similarly shine a light pulse to the ship once each year, as measured by clocks on earth. How often do the light pulses reach the spaceship?
- (e) Exactly half way to the planet, two of the astronauts get homesick and set off in a space module to return to earth. According to the astronauts who remain on the space ship, the module travels at a velocity of $\frac{5}{6}c$ in the direction toward earth. Find the total amount of time that the two astronauts will have been away according to people on earth.

Problem 5 (30 points)

Consider two infinite walls with perfect electrical conductivity. One wall is positioned in the x-y plane at $z = 0$, and the other wall is parallel to this wall and located at $z = L$. Recall from E&M that the component of an electric field parallel to a conducting surface must go to zero at the surface.

- (a) Write down expressions for the three lowest-frequency electromagnetic standing waves polarized along the x-direction that can be created in the space between the walls. What are the frequencies of these modes? (Assume the space between the walls is vacuum.)
- (b) Using your result from (a) and Maxwell's Equations, determine the magnetic field for the lowest-frequency standing wave.
- (c) Again determine the magnetic field for the lowest-frequency standing wave, this time using your result from (a) along with the fact that a standing wave can be considered the superposition of two traveling waves. Hint: you might find the following trig identities helpful:

$$\begin{aligned}\sin(x \pm y) &= \sin(x)\cos(y) \pm \cos(x)\sin(y) \\ \cos(x \pm y) &= \cos(x)\cos(y) \mp \sin(x)\sin(y)\end{aligned}$$

- (d) For the lowest-frequency standing wave, the electromagnetic energy flux that passes through an imaginary planes at $x = L/4$ and $x = L/2$ as a function of time. (The units of energy flux are $\frac{\text{energy}}{\text{area} \cdot \text{time}}$.)

Problem 6 (45 points)

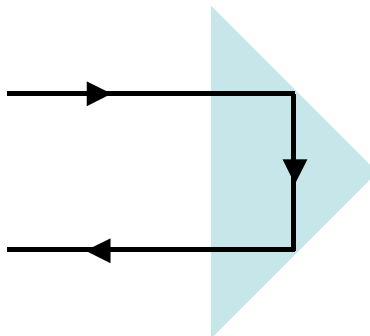
In each of the following five short problems, be sure to show your work and/or explain your answer in order to earn full credit.

- (a) **(10 points)** Consider an electromagnetic plane wave traveling in vacuum in the z -direction with wave vector k . The wave is linearly polarized along the direction at a 45-degree angle to the x - and y -axes, so that the electric field component can be written as:

$$\vec{E}(z,t) = E_0 \cos(\omega t - kz)(\hat{x} + \hat{y})$$

The wave is incident on a birefringent material of thickness L . A birefringent material is one that has two different indices of refraction for light polarized along different directions. In this case, the index of refraction is n_e for light polarized along \hat{x} and n_o for light polarized along \hat{y} , with $n_e > n_o$. One application of birefringent materials is in altering the polarization of light. Determine the thickness L that will turn this linearly polarized wave into a circularly polarized wave. (A piece of birefringent material that does this is called a quarter-wave plate.)

- (b) **(7 points)** Consider a piece of glass cut into an isosceles right triangle and surrounded on all sides by vacuum. A beam of light enters the glass through the base of the triangle with normal incidence, as shown in the picture below. The light undergoes total internal reflection twice inside of the glass so that all of the light ultimately returns in the direction from which it came. What is the minimum value for the index of refraction of the glass that will make this possible?



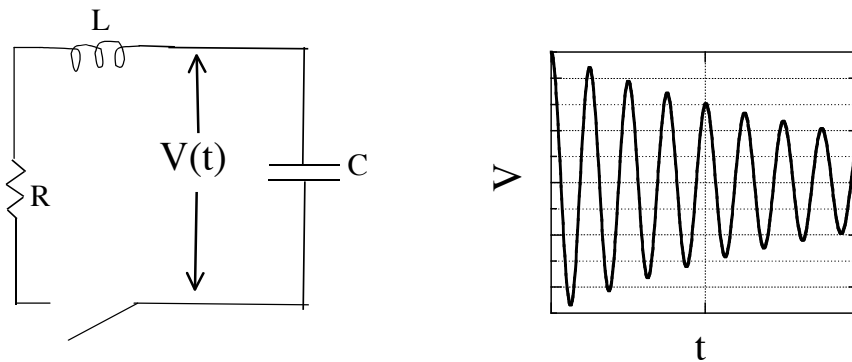
Problem 6 (continued)

- (c) **(8 points)** As mentioned in class, the dispersion relationship for a real piano string is actually somewhat different than the ideal case we derived for a string. More accurately, the frequency and wave vector for a piano string are related in the following way:

$$\omega^2 = \alpha k^2 + \beta k^4$$

What are the phase velocity and group velocity of a piano string that obeys this relationship? Give a one-sentence explanation for the physical meaning of each of these velocities.

- (d) **(10 points)** Consider the circuit below composed of a resistor R , inductor L , and capacitor C in series. Initially, the capacitor is charged and the switch is open. The plot on the right shows the voltage across the capacitor as a function of time after the switch is closed. We would like to increase the value of the resistor, keeping L and C fixed, in order to make the circuit overdamped. In terms of R , approximately how large must the new resistance be so that the circuit becomes overdamped? (You may find these approximations helpful: $e^{-0.5} \approx 0.6$ and $8\pi \approx 25$.)



- (e) **(10 points)** The wire entering my house for cable television is a transmission line with a characteristic impedance of 75Ω . When I attach wire to my TV, the TV terminates the line with an equivalent impedance of 50Ω . If a voltage pulse is sent down the line to my TV, what fraction of the pulse's energy is reflected from the connection back up the line?