171.201 FINAL EXAM FALL, 2006

Problem # 1

$$y(x,0) = \begin{cases} \frac{2hx}{L} & x < L/2 \\ 2h(1-x/L) & x > L/2 \end{cases}, \quad \dot{y}(x,0) = 0 \quad y(x,t) = \sum_{n=1} B_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t + \varphi_n)$$

(a) B_n w/ n = even are zero, by symmetry about the center point ($L/2$), or can be seen from the integral w/o performing it.

$$B_n = \frac{2}{L}\left[\int_0^{L/2} \frac{2hx}{L}\sin\frac{\omega_n x}{c}\,dx + \int_{L/2}^{L} 2h(1-x/L)\sin\frac{\omega_n x}{c}\,dx\right] \quad \text{w/ } \omega_n = \frac{n\pi c}{L}$$

(b) $\varphi_n = 0$ because of boundary conditions at $L = x$ and $x = 0$

(c) from part (a)

$$B_n = \frac{2}{L}\left[\int_0^{L/2} \frac{2hx}{L}\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} 2h(1-x/L)\sin\frac{n\pi x}{L}\,dx\right]$$

$$= \frac{2}{L}\left[\frac{2h}{L}\int_0^{L/2} x\sin\frac{n\pi x}{L}\,dx + 2h\int_{L/2}^{L}\sin\frac{n\pi x}{L}\,dx - \frac{2h}{L}\int_{L/2}^{L} x\sin\frac{n\pi x}{L}\,dx\right]$$

$$u = \frac{n\pi x}{L} \Rightarrow x = \frac{uL}{n\pi} \qquad du = \frac{n\pi}{L}dx \Rightarrow dx = \frac{L}{n\pi}du$$

$$\int_0^{L/2} x\sin\frac{n\pi x}{L}\,dx \longrightarrow \int_0^{\frac{n\pi}{2}} \frac{uL^2}{n^2\pi^2}\sin u\,du = \frac{L^2}{n^2\pi^2}\int_0^{\frac{n\pi}{2}} u\sin u\,du = \frac{L^2}{n^2\pi^2}\left(-u\cos u + \sin u\right)\Big|_0^{n\pi/2}$$

$$= \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\int_{L/2}^{L} x\sin\frac{n\pi x}{L} \longrightarrow \frac{L^2}{n^2\pi^2}\left(-u\cos u + \sin u\right)\Big|_{\frac{n\pi}{2}}^{n\pi} = \frac{L^2}{n^2\pi^2}\left(-n\pi(-1)^n + \sin\frac{n\pi}{2}\right)$$

$$\int_{n\pi/2}^{n\pi} \sin u\,du = -\cos u\Big|_{n\pi/2}^{n\pi} = +n\pi(-1)^n$$

adding this up, we get $\left[\frac{4L^2}{(n\pi)^2}\sin\frac{n\pi}{2} + \frac{4L^2}{(n\pi)^2}\sin\frac{n\pi}{2}\right]\frac{h}{L^2}$

$$\Rightarrow I = \frac{8h}{n^2\pi^2}$$

(d) $E_{pot} = \frac{1}{2}T\int\left(\frac{\partial y}{\partial x}\right)^2 dx \qquad \frac{\partial y}{\partial x} = \begin{cases} \frac{2h}{L} & 0 \le x \le L/2 \\ \frac{2h}{L} & \frac{L}{2} \le x \le L \end{cases} \Rightarrow E_{pot} = \frac{1}{2}T\int_0^L\left(\frac{2h}{L}\right)^2 dx = \frac{1}{2}\frac{T4h^2L}{L^2} = \frac{2Th^2}{L}$

(e) $E_n = \frac{1}{4}\rho L\omega_n^2 B_n^2 = \frac{1}{4}m\omega_n^2 B_n^2 = \frac{1}{4}\omega_n^2\frac{64h^2}{n^4\pi^4} = \frac{16h^2m}{\pi^2}\frac{\omega_n^2}{n^4} = \frac{16h^2m}{\pi^4}\frac{n^2\pi^2c^2}{L^2n^4} = \frac{16h^2mc^2}{\pi^2L^2}\frac{1}{n^2}$

$$E = \sum_{n,\,odd} E_n = \frac{16c^2h^2m}{\pi^2L^2}\sum_{n\,odd}\frac{1}{n^2} = \frac{16c^2h^2m}{\pi^2L^2}\frac{\pi^2}{8} = \frac{2c^2h^2m}{L^2} = \frac{2(T/\rho)h^2L\rho}{L^2} = \frac{2Th^2}{L} \text{ as before.}$$

Problem #2

(a) By symmetry, the waves from S_1 & S_3 arrive in phase with each other. The wave from S_2 travels a path length that is longer by b. Therefore, it will acquire an additional phase $\phi = \frac{2\pi}{\lambda} b$. Thus, at R the total $\bar{E}$ field will be:

$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 = E_0 \cos\omega t + E_0 \cos\omega t + E_0 \cos(\omega t + \phi) = 2E_0 \cos\omega t + E_0 \cos(\omega t + \phi)$$

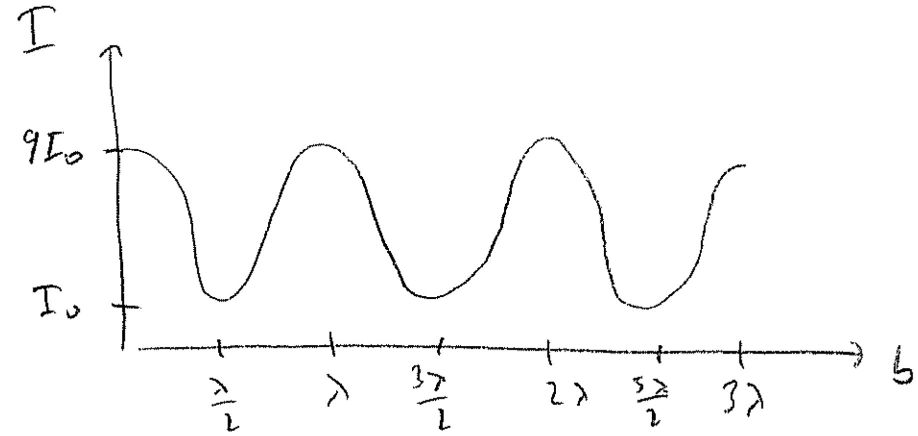
$$I \sim \langle \bar{E}^2 \rangle = \langle 4E_0^2 \cos^2\omega t + E_0^2 \cos^2(\omega t + \phi) + 4E_0^2 \cos\omega t \cos(\omega t + \phi) \rangle$$

$$= \langle 4E_0^2 \cos^2\omega t + E_0^2 \cos^2(\omega t + \phi) + 4E_0^2 \cos\omega t (\cos\omega t \cos\phi - \sin\omega t \sin\phi) \rangle$$

but, $\langle \cos^2\omega t \rangle = \frac{1}{2}$ & $\langle \cos\omega t \sin\omega t \rangle = 0$

$$\Rightarrow I = 4I_0 + I_0 + 4I_0 \cos\phi = \boxed{5I_0 + 4I_0 \cos\phi \quad \text{where } \phi = \frac{2\pi}{\lambda} b}$$

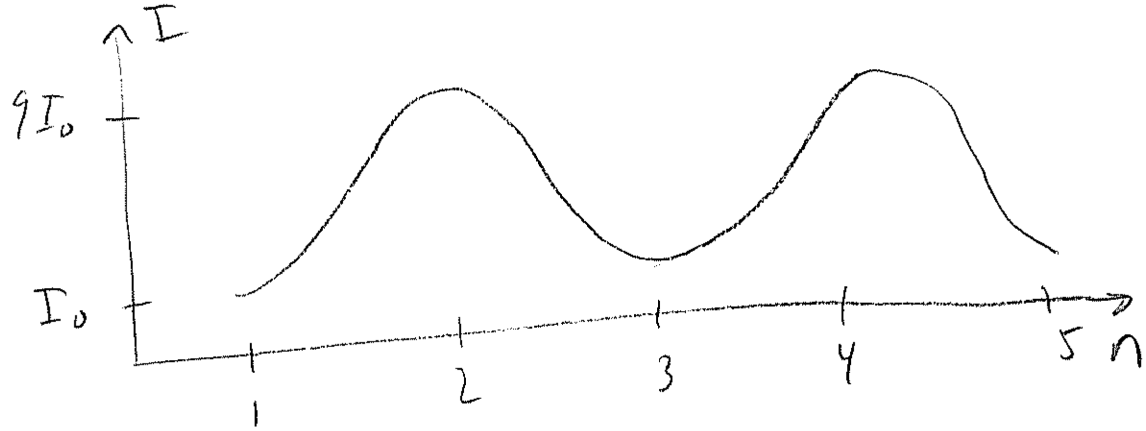
(b)



(c) On passing through the material of thickness b, the wave from S_2 picks up a phase $\phi = \frac{2\pi n b}{\lambda}$. Otherwise, the path length is the same for S_2 as for S_1 + S_3. So, the result is the same as in pt. (a)

$$I = 5 I_0 + 4 I_0 \cos(\pi n)$$

and plot looks the same



Prob.3

(a)

$$m\ddot{x} + mg\tan\theta = 0$$

if θ is small, $\tan\theta \simeq \frac{x}{l}$, then EOM can be re-written as:

$$\boxed{\ddot{x} + \frac{g}{l}x = 0}$$

(b)

$$\begin{cases} m\ddot{x}_1 = -mg\frac{x_1}{l} - k(x_1 - x_2) \\ m\ddot{x}_2 = -mg\frac{x_2}{l} + k(x_1 - x_2) \end{cases} \longrightarrow \begin{cases} \ddot{x}_1 + \frac{g}{l}x_1 + \frac{k}{m}(x_1 - x_2) = 0 & ① \\ \ddot{x}_2 + \frac{g}{l}x_2 - \frac{k}{m}(x_1 - x_2) = 0 & ② \end{cases}$$

(c)

let $X = x_1 + x_2$, then ① + ②

$$\ddot{X} + \frac{g}{l}X = 0$$

$$\ddot{X} + \omega_1^2 X = 0, \text{ where } \boxed{\omega_1^2 = \frac{g}{l}}$$

let $Y = x_1 - x_2$, then ① − ②

$$\ddot{Y} + \left(\frac{g}{l} + \frac{2k}{m}\right)Y = 0$$

$$\ddot{Y} + \omega_2^2 Y = 0, \text{ where } \boxed{\omega_2^2 = \frac{g}{l} + \frac{2k}{m}}$$

(d) let

$X(t) = x_1(t) + x_2(t) = A\cos(\omega_1 t + \phi_1)$, $Y(t) = x_1(t) - x_2(t) = B\cos(\omega_2 t + \phi_2)$

Then $x_1(t) = \frac{X(t) + Y(t)}{2} = \frac{1}{2}[A\cos(\omega_1 t + \phi_1) + B\cos(\omega_2 t + \phi_2)]$, $x_2(t) = \frac{X(t) - Y(t)}{2} = \frac{1}{2}[A\cos(\omega_1 t + \phi_1) - B\cos(\omega_2 t + \phi_2)]$

According to the initial conditions $x_1(0) = 0$, $\dot{x}_1(0) = v_0$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$, we get

$\phi_1 = \phi_2 = -\frac{\pi}{2}$ (for convenience)

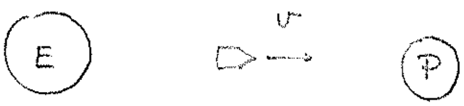
$$\begin{cases} A\omega_1 + B\omega_2 = 2v_0 \\ A\omega_1 - B\omega_2 = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{v_0}{\omega_1} \\ B = \frac{v_0}{\omega_2} \end{cases}$$

$$\boxed{\frac{A}{B} = \frac{\omega_2}{\omega_1} = \sqrt{1 + \frac{2kl}{mg}}}$$

Special Relativity & Waves

<u>Final Exam</u>

Problem 4



$d = 40 \text{ years} \times c$

a) The time for the astronaut to reach the planet, observed by an observer on earth, is

$$\Delta t = \frac{d}{v}$$

This should be related to the time measured by the astronaut by the time dilation relation.

$$\Delta t = \gamma \, \Delta t'$$

Combining the two above, we have

$$\frac{d}{v} = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}$$

$$1 - \frac{v^2}{c^2} = \frac{v^2 \Delta t'^2}{d^2}$$

$$1 = v^2 \left(\frac{1}{c^2} + \frac{\Delta t'^2}{d^2} \right)$$

$$v = \frac{1}{\left(\frac{1}{c^2} + \frac{\Delta t'^2}{d^2} \right)^{1/2}} = \frac{1}{\left(\frac{1}{c^2} + \left(\frac{20 \text{ yrs}}{40 \text{ yrs} \times c} \right)^2 \right)^{1/2}}$$

$$= \frac{1}{\left(\frac{1}{c^2} + \frac{9}{16 c^2} \right)^{1/2}} = \left(\frac{16}{25} c^2 \right)^{1/2} = \frac{4}{5} c$$

$$\therefore \boxed{v = \frac{4}{5} c}$$

Problem 4, continued....

b)

$$d' = \frac{d}{\gamma} = d\sqrt{1 - \frac{v^2}{c^2}} = d\sqrt{1 - \frac{16}{25}}$$

$$= \frac{3}{5} d = \frac{3}{5} \times 40 \text{ lyrs} = 24 \text{ lyrs}$$

$$\therefore \boxed{24 \text{ light years}}$$

c) Since the source, i.e. spaceship, is moving away from the observer

$$\nu = \nu' \sqrt{\frac{c - v}{c + v}} = \nu' \sqrt{\frac{c - \frac{4}{5}c}{c + \frac{4}{5}c}}$$

$$= \nu' \sqrt{\frac{\frac{\cancel{c}}{5}}{\frac{9}{5}\cancel{c}}} = \frac{1}{3}\nu'$$

We are given the period, T, of the signal. $\nu = \frac{1}{T}$

$$\frac{1}{T} = \frac{1}{3}\frac{1}{T'}$$

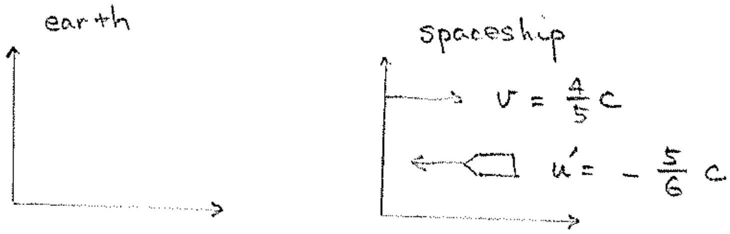
$$T = 3T' = 3 \times 1 \text{ year} = \boxed{3 \text{ years}}$$

d) By symmetry (earth is the source moving away from the observer, spaceship)

$$\boxed{T = 3 \text{ years.}}$$

Problem 4, continued....

e)



Using relativistic velocity addition theorem

$$u = \frac{u' + v}{1 + u'v/c^2} = \frac{-\frac{5}{6}c + \frac{4}{5}c}{1 + \left(-\frac{5}{6}\right)\left(\frac{4}{5}\right)} = \frac{-\frac{1}{30}c}{\frac{1}{3}} = -\frac{c}{10}$$

The total amount of time according to people on earth is

$$t = \frac{d}{2}\cdot\frac{1}{v} + \frac{d}{2}\cdot\frac{1}{u}$$

$$= \frac{40\text{ yrs} \times c}{2 \times \frac{4}{5}c} + \frac{40\text{ yrs} \times c}{2 \times \frac{1}{10}c} = 25\text{ yrs} + 200\text{ yrs}$$

$$= \boxed{225\text{ years}}$$

Problem # 5

(a) Boundary conditions $\bar{E}(0,t) = \bar{E}(L,t) = 0$

$\Rightarrow$ standing wave solutions:

$$\bar{E}(z,t) = E_0 \sin(k_n z)\sin(\omega_n t)\,\hat{x}$$

where $k_n = \frac{n\pi}{L}$ $\quad n$ = integer

frequency: $\omega_n = c k_n \Rightarrow \omega_n = \frac{n c \pi}{L}$

(b) $\bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t}$

$$\bar{\nabla}\times\bar{E} = \hat{x}\left(\cancel{\frac{\partial E_z}{\partial y}}^{\,0} - \cancel{\frac{\partial E_y}{\partial z}}^{\,0}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \cancel{\frac{\partial E_z}{\partial x}}^{\,0}\right) + \hat{z}\left(\cancel{\frac{\partial E_y}{\partial x}}^{\,0} - \cancel{\frac{\partial E_x}{\partial y}}^{\,0}\right)$$

$$\Rightarrow -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} = E_0 k_n \cos(k_n z)\sin(\omega_n t)\,\hat{y}$$

$$\Rightarrow \bar{B}(z,t) = \underbrace{\frac{c k_n}{\omega_n}}_{=1} E_0 \cos(k_n z)\cos(\omega_n t)\,\hat{y}$$

$$\boxed{\bar{B}(z,t) = E_0 \cos(k_n z)\cos(\omega_n t)\,\hat{y}}$$

(c) $\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$

$$\Rightarrow \bar{E}(z,t) = \frac{E_0}{2}\underbrace{\left(\cos(k_n z - \omega_n t) - \cos(k_n z + \omega_n t)\right)}_{\text{2 traveling waves going in } \pm\hat{z} \text{ direction.}}\hat{x}$$

But, for traveling waves, $\bar{E}$ & $\bar{B}$ are in phase & $\hat{E}\times\hat{B} = \hat{k}$

$\longrightarrow$

5(c), continued.

$$\Rightarrow \vec{B}(z,t) = \frac{E_0}{2}\left(\cos(k_n z - \omega_n t) + \cos(k_n z + \omega_n t)\right)\hat{y}$$

but $\cos(x-y) + \cos(x+y) = 2\cos x \cos y$

$$\Rightarrow \boxed{\bar{B}(z,t) = E_0 \cos(k_n z)\cos(\omega_n t)\,\hat{y}}$$

(d) Energy flux given by Poynting vector:

$$\bar{S} = \frac{c}{4\pi}\bar{E}\times\bar{B}$$

$$= \frac{c}{4\pi}E_0^2 \sin(k_n x)\cos(k_n x)\sin(\omega_n t)\cos(\omega_n t)\,\hat{z}$$

$$n=1 \Rightarrow k_n = \pi/L$$

$z = L/4$:

$$\bar{S} = \frac{c}{4\pi}E_0\sqrt{2}\sqrt{2}\;\sin(\omega_n t)\cos(\omega_n t)\,\hat{z}$$

$$\boxed{\bar{S} = \frac{c}{2\pi}E_0 \sin\omega_n t\cos\omega_n t\;\hat{z}}$$

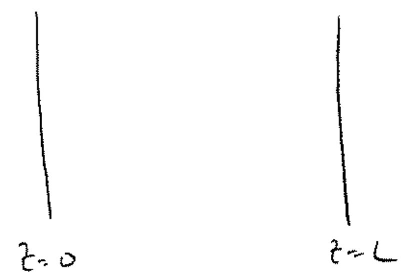
$z = L/2$

$$\bar{S} = \frac{c}{4\pi}E_0 \sin\left(\frac{\pi}{2}\right)\underbrace{\cancel{\cos\left(\frac{\pi}{2}\right)}}_{0}\sin\omega_n t\cos\omega_n t\;\hat{z}$$

$$\boxed{\bar{S} = 0}$$

Problem # 6

(a) $\vec{E}(z,t) = E_0 \cos(\omega t - kz)\hat{x} + E_0 \cos(\omega t - kz)\hat{y}$



@ $z=0$ $\quad \vec{E}(z,t) = E_0 \cos(\omega t)\hat{x} + E_0 \cos(\omega t)\hat{y}$

@ $z=L$ $\quad \vec{E}(z,t) = E_0 \cos(\omega t - k_x L)\hat{x} + E_0 \cos(\omega t - k_y L)\hat{y}$

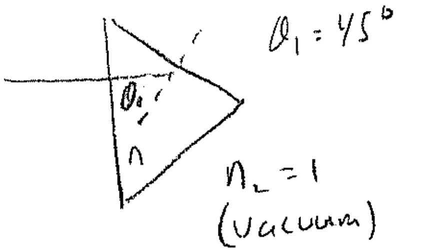
For circularly polarized, $\hat{x}$ & $\hat{y}$ components should be out of phase $\Longrightarrow$

$$k_x L - k_y L = \frac{(2m+1)\pi}{4} \quad \text{where } m = \text{integer}$$

$$\frac{2\cancel{\pi} n_e}{\lambda} L - \frac{2\cancel{\pi} n_o}{\lambda} L = \frac{(2m+1)\cancel{\pi}}{4}$$

$$\boxed{L = \frac{(2m+1)}{4} \frac{\lambda}{n_e - n_o}}$$

(b) Snell's Law : $\quad \dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{n_2}{n_1}$



For total internal reflection $\sin\theta_2 \geq 1$

$$\Rightarrow \quad \sin\theta_1 = \frac{1}{n}$$

$$\Rightarrow \quad n = (\sin 45°)^{-1} = \boxed{\sqrt{2}}$$

Prob. 6 (continued)

(c)

$\omega^2 = \alpha k^2 + \beta k^4$

$$\boxed{\begin{aligned} &\text{phase velocity} = \frac{\omega}{k} = \sqrt{\alpha + \beta k^2} \\ &\text{group velocity} = \frac{d\omega}{dk} = \frac{\alpha + 2\beta k^2}{\sqrt{\alpha+\beta k^2}} \end{aligned}}$$

(d)

$L\ddot{q} + R\dot{q} + \frac{q}{C} = 0$

$A(t) = A_0 e^{-\gamma t}$

at $t = 4T$, $\frac{A(t)}{A_0} = \frac{3}{5} = 0.6 \rightarrow 0.6 = e^{-\gamma 4T}$

$\gamma = \frac{R}{2L} = \frac{1}{8T} = \frac{1}{8}\frac{\omega'}{2\pi} = \frac{\omega'}{16\pi}$, where $\omega' = \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{1/2} \approx \left(\frac{1}{LC}\right)^{1/2}$ for small R

$R = \frac{2L}{16\pi}\cdot\left(\frac{1}{LC}\right)^{1/2} = \frac{1}{8\pi}\sqrt{\frac{L}{C}} \simeq \frac{1}{25}\sqrt{\frac{L}{C}}$

To be over-damped

$\frac{R'^2}{4L^2} \geq \frac{1}{LC} \rightarrow R' \geq 2\sqrt{\frac{L}{C}} = 50R$

e) $Z_1 = 75\Omega$, $Z_2 = 50\Omega$

Ratio of the pulse's reflected energy $= \left(\frac{Z_1 - Z_2}{Z_1+Z_2}\right)^2 = \left(\frac{75-50}{75+50}\right)^2$

$$\boxed{= 4\%}$$

Problem #5

(a)

$$\underline{E}(z,t) = E_0 \sin(k_n z)\sin(\omega_n t)\,\hat{x}$$

BCs: $E(L,t) = 0 \Rightarrow E_0 \sin(k_n L)\sin(\omega_n t)\hat{x} = 0 \Rightarrow \sin(k_n L) = 0 \Rightarrow k_n L = n\pi$
$\Rightarrow k_n = \frac{n\pi}{L}$

$$k_n = \frac{n\pi}{L} = \frac{2\pi}{\lambda_n} \Rightarrow \lambda_n = \frac{2L}{n}$$

$$\lambda_n f_n = c \Rightarrow f_n = \frac{cn}{2L} \Rightarrow \omega_n = \frac{n\pi c}{L} \quad \text{(as in problem \#1)}$$

$$\omega_1 = \frac{\pi c}{L}$$

$$\omega_3 = \frac{3\pi c}{L}$$

$$\omega_5 = \frac{5\pi c}{L}$$

(b)

$$\underline{\nabla}\times\underline{E} = -\frac{\partial B}{\partial t} \qquad \underline{\nabla}\times\underline{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ \sim & 0 & 0 \end{vmatrix} = (0-0)\hat{x} - (0-\partial_z \sim)\hat{y} - (0-0)\hat{z} = \frac{\partial E_x}{\partial z}$$

$$= k_n E_0 \cos(k_n z)\sin(\omega_n t)\,\hat{y}$$

$$-\frac{\partial B}{\partial t} = \underline{\nabla}\times\underline{E} \Rightarrow B = -\int \underline{\nabla}\times\underline{E}\,dt = -k_n E_0 \cos(k_n z)\int \sin(\omega_n t)\,dt = E_0 \frac{k_n}{\omega_n}\cos(k_n z)\cos(\omega_n t)$$

$$\underline{B} = \frac{E_0}{c}\cos(k_n z)\cos(\omega_n t)\,\hat{y}$$

$$\omega_1 = \frac{\pi c}{L}$$

(c)

$$E = E_0 \sin(k_n z)\sin(\omega_n t) = \frac{1}{2}\left[\cos(k_n z - \omega_n t) - \cos(k_n z + \omega_n t)\right]$$

if $E = E_0 \cos(kz + \omega t)\hat{x}$

then $B = -E_0 \cos(kz - \omega t)\hat{y} \Rightarrow B = -\frac{1}{2}\left[\cos(k_n z - \omega t) - \cos(k_n z + \omega_n t)\right] = \frac{1}{2}\left[\cos(k_n z + \omega_n t) - \cos(k_n z - \omega_n t)\right]$

$$= E_0 \cos(k_n z)\cos(\omega_n t)$$

(d) $S = \underbrace{\frac{c}{4\pi}\frac{1}{\mu}}_{\alpha}\underline{E}\times\underline{B} = \alpha\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sim & 0 & 0 \\ 0 & \sim & 0 \end{vmatrix} = \alpha\left[(0-0)\hat{x} - (0-0)\hat{y} + (E_x B_y - 0)\hat{z}\right] \Rightarrow S = \alpha E_0^2 \sin k_n z \cos k_n z\,\overbrace{\sin\omega_n t\cos\omega_n t}^{f(t)}$

for $n=1$:

$$S(L/4) = \alpha E_0^2 \sin\left(\frac{\pi}{L}\frac{L}{4}\right)\cos\left(\frac{\pi}{L}\frac{L}{4}\right)f(t) = \frac{1}{2}\alpha E_0^2 f(t) = \frac{c}{8\pi\mu}E_0^2 f(t)$$

$$S(L/2) = \alpha E_0^2 \sin\left(\frac{\pi}{L}\frac{L}{2}\right)\cos\left(\frac{\pi}{L}\frac{L}{2}\right)f(t) = 0$$