171 201, Fall 2004

Midterm #2

(1)

(a) incident wave : $A\cos(\omega t - kx)$ $A = 1$ meter

reflected wave : $AR\cos(\omega t - kx)$

transmitted wave: $AT\cos(\omega t - kx)$

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} \qquad Z = \sqrt{T\rho} \Rightarrow Z_1 = \sqrt{16} = 4$$
$$Z_2 = \sqrt{25} = 5$$

$$\Rightarrow R = \frac{4-5}{4+5} = -\frac{1}{9}$$

$\Rightarrow$ amplitude of reflected wave = $\boxed{\frac{1}{9} \text{ meters}}$

$$T = 1 + R = 1 - \frac{1}{9} = \frac{8}{9}$$

$\Rightarrow$ amplitude of transmitted wave = $\boxed{\frac{8}{9} \text{ meters}}$

(b) continuity requires that frequencies for $x<0$ & $x>0$ be the same. $\Rightarrow$ $\boxed{\omega = 10 \text{ radians/second}}$

$$k = \frac{\omega}{v}, \quad v = \sqrt{\frac{T}{\rho}} = \sqrt{400} = 20 \text{ meters/second}$$

$$\Rightarrow k = \frac{10}{20} = \frac{1}{2}\ m^{-1}$$

$$\lambda = \frac{2\pi}{k} = \boxed{4\pi \text{ meters}}$$

(1)(c)

$$\text{Energy} \sim Z\,(\text{Amplitude})^2$$

$$\frac{\text{Reflected Energy}}{\text{Incident Energy}} = \left(\frac{1}{9}\right)^2 = \frac{1}{81}$$

$$(\text{Incident Energy}) = (\text{Transmitted Energy}) + (\text{Reflected Energy})$$

$$\Rightarrow \frac{\text{Reflected Energy}}{\text{Incident Energy}} + \frac{\text{Transmitted Energy}}{\text{Incident Energy}} = 1$$

$$\Rightarrow \frac{\text{Transmitted Energy}}{\text{Incident Energy}} = \frac{80}{81}$$

Alternatively,

$$\frac{\text{Transmitted Energy}}{\text{Incident Energy}} = \frac{Z_2}{Z_1}\left(\frac{8}{9}\right)^2 = \frac{5}{4}\cdot\frac{64}{81} = \frac{80}{81}$$

② (a) $M\ddot{x} + b\dot{x} = 0$

$x(t) = C - \frac{v_0}{\gamma}\exp(-\gamma t)$ a solution?

plug into eq. of motion:

$$M \cancel{\frac{v_0}{\gamma}}\gamma^{\cancel{2}}\cancel{e^{-\gamma t}} - \cancel{\frac{v_0}{\gamma}}\, b\cancel{\gamma}\cancel{e^{-\gamma t}} = 0$$

so, it is a solution with $\boxed{\gamma = \frac{b}{M}}$

(b) $m\ddot{x} + b\dot{x} = F_0\cos(\omega t)$

solution using complex analysis:

$M\ddot{z} + b\dot{z} = F_0 e^{i\omega t}$ $\qquad x = \mathrm{Re}(z)$

$z = Ae^{i(\omega t - \phi)} + R$

$\dot{z} = i\omega A e^{i(\omega t - \phi)}$

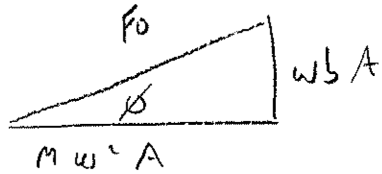
$\ddot{z} = -\omega^2 A e^{i(\omega t - \phi)}$

$\Downarrow$

$$-M\omega^2 A e^{i(\omega t - \phi)} + i\omega b A e^{i(\omega t - \phi)} = F_0 e^{i\omega t}$$

$$\phi = \tan^{-1}\left(\frac{\cancel{\omega} b \cancel{A}}{\omega^{\cancel{2}}\cancel{A}}\right)$$

$$\boxed{\phi = \tan^{-1}\left(\frac{b}{m\omega}\right)}$$



$$F_0^2 = (m\omega^2 A)^2 + (\omega b A)^2 = A^2(M^2\omega^4 + \omega^2 b^2)$$

$$\boxed{A = \frac{F_0}{\sqrt{M^2\omega^4 + \omega^2 b^2}} = \frac{F_0}{\omega\sqrt{M^2\omega^2 + b^2}}}$$

(2) (b) continued:

Solution using trig:

$$x = A\cos(\omega t - \phi) + R$$

$$\dot{x} = -\omega A\sin(\omega t - \phi)$$

$$\ddot{x} = -\omega^2 A\cos(\omega t - \phi)$$

$$\Rightarrow -m\omega^2 A\cos(\omega t - \phi) - b\omega A\sin(\omega t - \phi) = F_0\cos(\omega t)$$

$$-m\omega^2 A\left[\cos\omega t\cos\phi + \sin\omega t\sin\phi\right] - b\omega A\left\{\sin\omega t\cos\phi - \cos\omega t\sin\phi\right\} = F_0\cos\omega t$$

$\sin\omega t$ terms must add to 0

$$-m\omega^2 A\sin\phi - b\omega A\cos\phi = 0$$

$$\Rightarrow \tan\phi = \frac{b\omega A}{m\omega^2 A} = \boxed{\frac{b}{m\omega}}$$

$\cos\omega t$ terms must add to F_0:

$$-m\omega^2 A\cos\phi - b\omega A\sin\phi = F_0$$

$$-m\omega^2 A - b\omega A\tan\phi = F_0/\cos\phi \quad \text{but} \quad \frac{1}{\cos\phi} = (\tan^2\phi + 1)^{1/2}$$

$$-m\omega^2 A - b\cancel{\omega} A\, b/m\cancel{\omega} = F_0\left(\frac{b^2}{m^2\omega^2} + 1\right)^{1/2}$$

$$A\left(m\omega^2 + \frac{b^2}{m}\right) = F_0\left(\frac{b^2}{m^2\omega^2} + 1\right)^{1/2}$$

$$A = \frac{F_0\left(\frac{b^2}{m^2\omega^2} + 1\right)^{1/2}}{\frac{1}{m}(\omega^2 m^2 + b^2)} = \frac{F_0\frac{1}{m\omega}(b^2 + m^2\omega^2)^{1/2}}{\frac{1}{m}(\omega^2 m^2 + b^2)}$$

$$\boxed{A = \frac{F_0}{\omega\left(\omega^2 m^2 + b^2\right)^{1/2}}}$$

② (c) general solution = steady state + transient

$$x(t) = A\cos(\omega t - \phi) + C - \frac{V_0}{\gamma}\exp(-\gamma t)$$

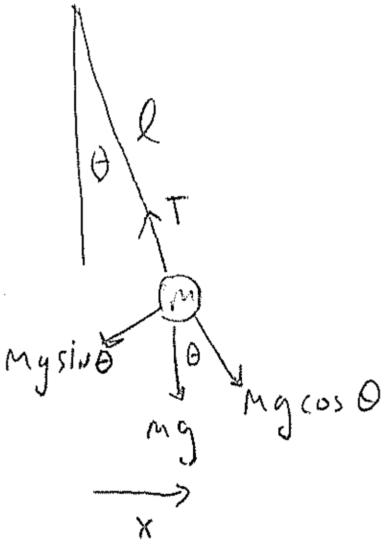
$$\dot{x}(t) = -\omega A \sin(\omega t - \phi) + V_0 \exp(-\gamma t)$$

$$x(0) = 0 = A\cos\phi + C - \frac{V_0}{\gamma}$$

$$\dot{x}(0) = 0 = \omega A \sin\phi + V_0 \Rightarrow \boxed{V_0 = -\omega A \sin\phi}$$

$$\boxed{C = \frac{V_0}{\gamma} - A\cos\phi}$$

(3) (a)



$$\sum \tau = I\ddot{\theta}$$

$$-Mgl\sin\theta = Ml^2\ddot{\theta}$$

small angles

$$\theta \approx \sin\theta \approx \tan\theta = \frac{x}{l}$$

$$\Rightarrow \quad \ddot{\theta} \approx \ddot{x}/l$$

$$-Mgl\,\frac{x}{l} = Ml^2\,\frac{\ddot{x}}{l}$$

$$\Rightarrow \quad \boxed{\ddot{x} + \frac{g}{l}x = 0}$$

(b)

$$\boxed{\begin{aligned} &M\ddot{x}_1 + \frac{Mg x_1}{l} + kx_1 - k(x_2 - x_1) = 0 \\ &M\ddot{x}_2 + \frac{Mg x_2}{l} + kx_2 + k_2(x_2 - x_1) = 0 \end{aligned}}$$

(c) $u = x + y$

$v = x - y$

add equations of motion: $M(\ddot{x}_1 + \ddot{x}_2) + \left(\frac{Mg}{l} + k\right)(x_1 + x_2) = 0$

$$M\ddot{u} + \left(\frac{Mg}{l} + k\right)u = 0$$

$$\rightarrow u(t) = A_u \cos(\omega_u t + \phi_u)$$

with $\boxed{\omega_u^2 = \dfrac{\frac{Mg}{l} + k}{M} = \dfrac{g}{l} + \dfrac{k}{M}}$

③ (c) continued

subtract eqs. of motion:

$$m(\ddot{x}_1 - \ddot{x}_2) + \left(\frac{mg}{\ell} + k\right)(x_1 - x_2) + 2k(x_1 - x_2) = 0$$

$$m\ddot{v} + \left(\frac{mg}{\ell} + 3k\right)v = 0$$

$$\Rightarrow v(t) = A_v \cos(\omega_v t + \phi_v)$$

with $\boxed{\omega_v^2 = \frac{\frac{mg}{\ell} + 3k}{m} = \frac{g}{\ell} + \frac{3k}{m}}$

(d)

$$x_1(t) = \tfrac{1}{2}(u(t) + v(t))$$
$$= \tfrac{1}{2}\left(A_u \cos(\omega_u t + \phi_u) + A_v \cos(\omega_v t + \phi_v)\right)$$

$$x_2(t) = \tfrac{1}{2}(u(t) - v(t))$$
$$= \tfrac{1}{2}\left(A_u \cos(\omega_u t + \phi_u) - A_v \cos(\omega_v t + \phi_v)\right)$$

$$\dot{x}_1(t) = -\tfrac{1}{2}\left(\omega_u A_u \sin(\omega_u t + \phi_u) + \omega_v A_v \sin(\omega_v t + \phi_v)\right)$$

$$\dot{x}_2(t) = -\tfrac{1}{2}\left(\omega_u A_u \sin(\omega_u t + \phi_u) - \omega_v A_v \sin(\omega_v t + \phi_v)\right)$$

$$x_1(0) = 0 \Rightarrow A_u \cos\phi_u + A_v \cos\phi_v = 0$$
$$x_2(0) = 0 \Rightarrow A_u \cos\phi_u - A_v \cos\phi_v = 0$$
$$\Rightarrow \cos\phi_u = \cos\phi_v = 0 \Rightarrow \phi_u = \phi_v = \pi/2$$

$$\dot{x}_1(0) = v_0 = \omega_u A_u + \omega_v A_v$$

$$\dot{x}_2(0) = 0 = \omega_u A_u - \omega_v A_v \Rightarrow \boxed{\frac{A_u}{A_v} = \frac{\omega_v}{\omega_u}}$$