171.201 Fall 2005

Midterm #2, solutions

① (a) $m\ddot{x}_1 = -kx_1 + G(x_2 - x_1)$

$m\ddot{x}_2 = -kx_2 - G(x_2 - x_1)$

$$\Rightarrow \boxed{\begin{aligned} \ddot{x}_1 + \frac{k+G}{m}x_1 - \frac{G}{m}x_2 &= 0 \\ \ddot{x}_2 - \frac{G}{m}x_1 + \frac{k+G}{m}x_2 &= 0 \end{aligned}}$$

(b) define $u = x_1 + x_2$, $v = x_1 - x_2$

add eqs of motion:

$$\ddot{x}_1 + \ddot{x}_2 + \frac{k}{m}(x_1 + x_2) = 0$$

ie. $\ddot{u} + \frac{k}{m}u = 0 \quad \Longrightarrow \quad u = A_u \cos(\omega_u t + \phi_u)$

w/ $\boxed{\omega_u = \sqrt{k/m}}$

subtract eq. of motion:

$$\ddot{x}_1 - \ddot{x}_2 + \frac{k+2G}{m}(x_1 - x_2) = 0$$

ie. $\ddot{v} + \frac{k+2G}{m}v = 0 \quad \Longrightarrow \quad v = A_v \cos(\omega_v t + \phi_v)$

w/ $\boxed{\omega_v = \sqrt{\frac{k+2G}{m}}}$

(c) $x_1 = \frac{1}{2}(u+v)$

$$x_1(t) = \frac{A_u}{2}\cos(\omega_u t + \phi_u) + \frac{A_v}{2}\cos(\omega_v t + \phi)$$

$x_2 = \frac{1}{2}(u - v)$

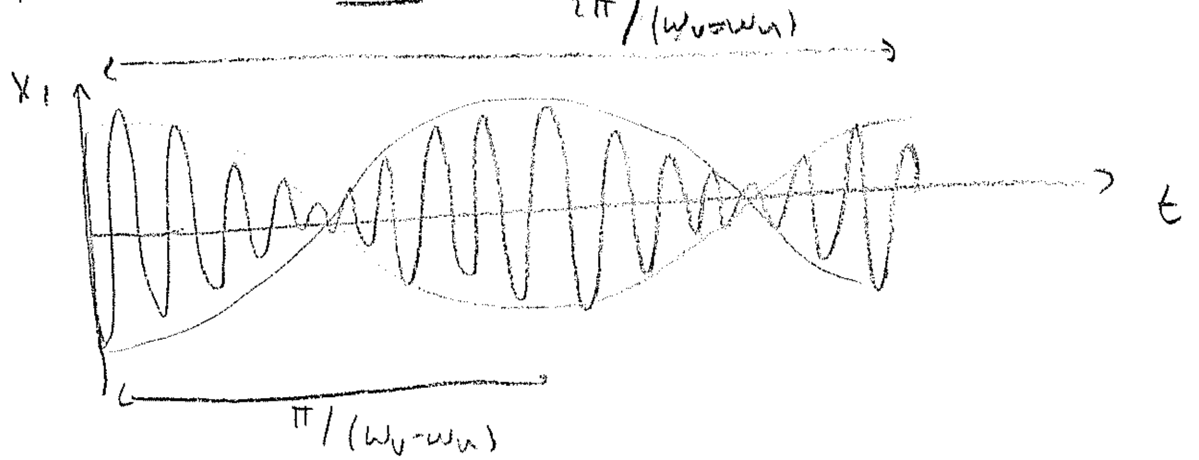
$$x_2(t) = \frac{A_u}{2}\cos(\omega_u t + \phi) - \frac{A_v}{2}\cos(\omega_v t + \phi)$$

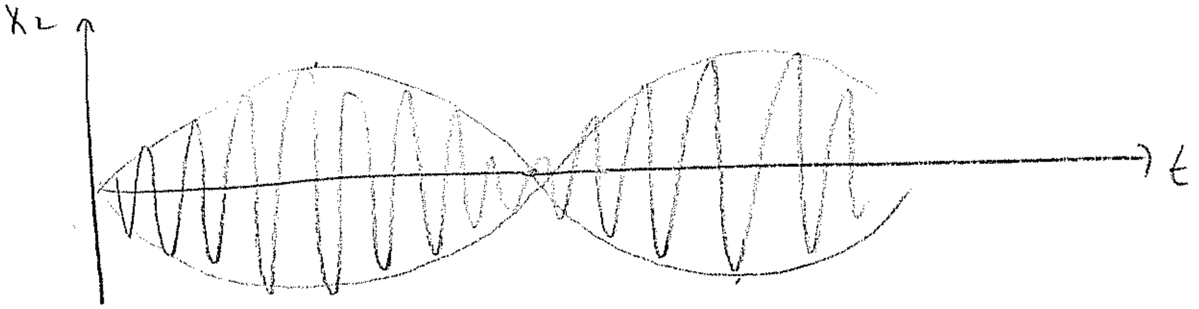
$$x_1(0) = \frac{A_u}{2}\cos\phi_u + \frac{A_v}{2}\cos\phi_v = x_0$$

$$x_2(0) = \frac{A_u}{2}\cos\phi_u - \frac{A_v}{2}\cos\phi_v = 0$$

① continued

So, x_1 (x_2) is the sum (difference) of two equal cosine terms with frequencies $\omega_u \approx \omega_v$ if $G \ll K$. Resulting motion is thus **beats**.





$$\Rightarrow \Delta t = \frac{\pi}{\omega_v-\omega_u} \quad \text{where} \quad \omega_v-\omega_u = \sqrt{\frac{K+2G}{m}} - \sqrt{\frac{K}{m}}$$

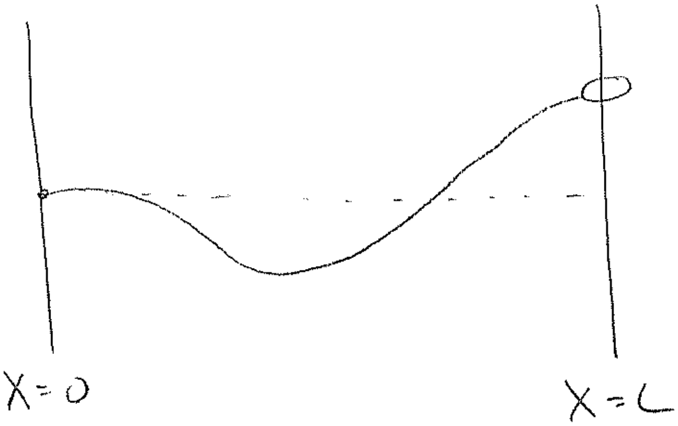
$$= \left(\frac{K}{m}\right)^{1/2}\left(1+\frac{2G}{K}\right)^{1/2} - \left(\frac{K}{m}\right)^{1/2}$$

$$\approx \left(\frac{K}{m}\right)^{1/2}\left(1+\frac{G}{K}\right) - \left(\frac{K}{m}\right)^{1/2}$$

$$\approx \frac{G}{\sqrt{Km}}$$

$$\Rightarrow \boxed{\Delta t \approx \frac{\pi\sqrt{Km}}{G}}$$

(2)



standing wave: $y(x,t) = A\cos\omega t \sin(kx+\phi)$

Boundary conditions

(1) $y(0,t)=0$

$\Rightarrow A\cos\omega t \sin\phi = 0 \Rightarrow \phi = 0$

(2) $\left.\frac{dy}{dx}\right|_{x=L} = 0$

$\Rightarrow Ak\cos\omega t \cos(kL) = 0$

$\Rightarrow kL = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots = \frac{(2n-1)\pi}{2}$

$n=1,2,3,4\ldots$

$\Rightarrow \lambda = \frac{2\pi}{k} = \frac{4L}{(2n-1)}$

$\boxed{\lambda_1 = 4L,\ \lambda_2 = \frac{4}{3}L,\ \lambda_3 = \frac{4}{5}L}$

$\omega = vk = \sqrt{\frac{T}{\rho}}k$ $\boxed{\omega_1 = \sqrt{\frac{T}{\rho}}\frac{\pi}{2L},\ \omega_2 = \sqrt{\frac{T}{\rho}}\frac{3\pi}{2L},\ \omega_3 = \sqrt{\frac{T}{\rho}}\frac{5\pi}{2L}}$

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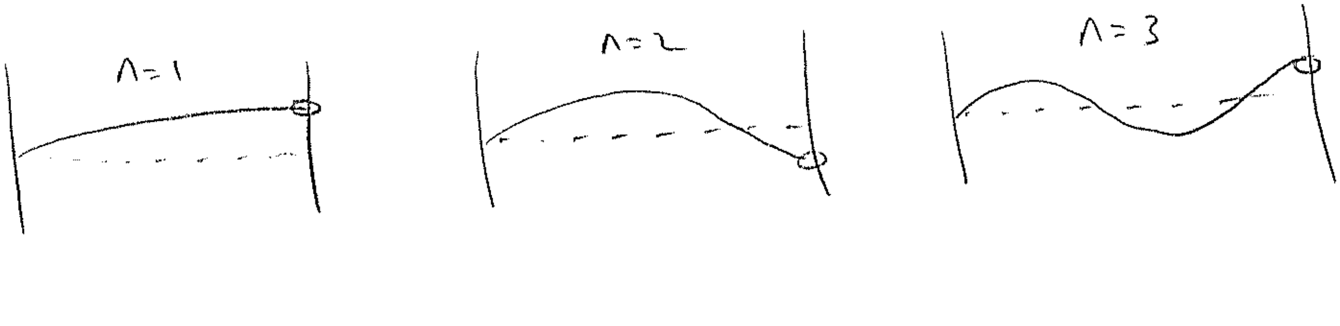

3

(a) $$Q^{-1} = \frac{FWHM}{w_0}$$

$$FWHM = (1010 - 990)\ s^{-1} = 20\ s^{-1}$$

$$w_0 = 1000\ s^{-1}$$

$$\Rightarrow Q = \frac{1000}{20} = \boxed{50}$$

(b) $P(w_0) = 20$ Watts $= 20$ Joules/sec

$$T = \frac{2\pi}{w_0} = \frac{2\pi}{1000} \text{ seconds}$$

work in one cycle: $$W = PT = 20\left(\frac{2\pi}{1000}\right) = \frac{4\pi}{100} = \boxed{\frac{\pi}{25} \text{ Joules}}$$

(c) $$E = E_0 e^{-\frac{w_0 t}{Q}}$$

$$\Rightarrow E = \frac{E_0}{e} \text{ @ } t = \frac{Q}{w_0} = \frac{50}{1000\ s^{-1}} = \boxed{0.05 \text{ seconds}}$$

4

(a) For string $F = -Z \frac{dy}{dt}$

If there is no reflection, then impedances are matched $\Rightarrow$ $C = Z = \boxed{\sqrt{T\rho}}$

(b) if $C = \frac{\sqrt{T\rho}}{2}$ then

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{\sqrt{T\rho} - \frac{\sqrt{T\rho}}{2}}{\sqrt{T\rho} + \frac{\sqrt{T\rho}}{2}} = \frac{1}{3}$$

$\Rightarrow$ Amplitude of reflected wave $= \boxed{A/3}$

(c) Power $\propto$ (amplitude)^2

$\Rightarrow$ reflected power $= \frac{1}{9}$ incident power

$\Rightarrow$ absorbed power $= \frac{8}{9}$ incident power

$$= \frac{8}{9}\left(\frac{1}{2} Z A^2 \omega^2\right)$$

$$= \boxed{\frac{4}{9}\sqrt{T\rho}\, A^2 \omega^2}$$