Midterm 2: Example Solutions

(1a) In the limit where the oscillations are small, we can make the following approximations:

(i) Tension in strings is the same as in equilibrium (2Mg and Mg for upper and lower strings respectively)
(ii) Motions are nearly horizontal
(iii) For any angle θ to the vertical, $\sin\theta \sim \tan\theta \sim \theta$

➔ Force exerted by upper string on mass A $= -2Mg\Psi_A/L$
Force exerted by lower string on mass A $= -Mg(\Psi_A - \Psi_B)/L$

Force exerted by lower string on mass B $= +Mg(\Psi_A - \Psi_B)/L$

Equations of motion: $\mathbf{d^2\Psi_A/dt^2 = (g/L)(\Psi_B - 3\Psi_A)}$
$\mathbf{d^2\Psi_B/dt^2 = (g/L)(\Psi_A - \Psi_B)}$

(b) Normal coordinates are of the form $\Psi_N = \Psi_A + r\Psi_B$

for which
$$d^2\Psi_N/dt^2 = d^2\Psi_A/dt^2 + r\, d^2\Psi_B/dt^2$$
$$= (g/L)(\Psi_B - 3\Psi_A + r\Psi_A - r\Psi_B)$$
$$= (g/L)([r-3]\Psi_A + [1-r]\Psi_B)$$

The right-hand-side is a multiple of Ψ_N (in fact, $[g/L][r-3]$ times Ψ_N) provided $(1-r)/(r-3) = r$.

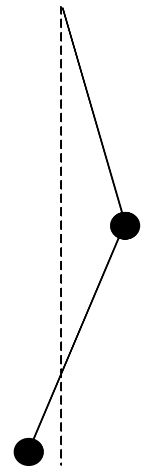
This requires $1 - r = r^2 - 3r \iff r^2 - 2r - 1 = 0 \iff r = 1 \pm \sqrt{2}$

Normal coordinates are $\Psi_1 = \Psi_A + (1+\sqrt{2})\Psi_B$
$\Psi_2 = \Psi_A + (1-\sqrt{2})\Psi_B$

Equation of motion is then $d^2\Psi_N/dt^2 = (g/L)[-2 \pm\sqrt{2}]\Psi_N$

Frequencies are $\sqrt{[(g/L)(2 \mp\sqrt{2})]}$

(c) Normal modes:

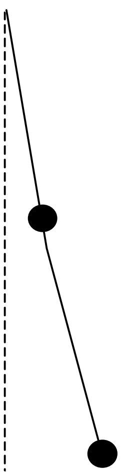


$\Psi_1 = 0 \implies \Psi_A = -(1 + \sqrt{2})\, \Psi_B$

Antisymmetric mode in which masses move in opposite directions.

Amplitude of mass A ~ 2.4 times amplitude of mass B

Frequency = $\sqrt{[(g/L)(2 + \sqrt{2})]}$



$\Psi_2 = 0 \implies \Psi_A = (\sqrt{2} - 1)\, \Psi_B$

Symmetric mode in which masses move in same directions
Amplitude of mass A ~ 0.4 times amplitude of mass B

Frequency = $\sqrt{[(g/L)(2 - \sqrt{2})]}$

(2a) Equation of motion: $\mathbf{d^2\Psi/dt^2 + \Gamma\, d\Psi/dt + \omega_0^2\, \Psi = F_0 \cos \omega_0 t}$
where $\omega_0 = \sqrt{(K/M)}$

(b) The transient solutions to the equation die away on a timescale $1/\Gamma$, leaving us with the steady-state solution for time, $t >$ few x $1/\Gamma$

(c) Try solution of form $\Psi = \Psi_0 \sin \omega_0 t$
(for which $d\Psi/dt = \omega_0 \Psi_0 \cos \omega_0 t$ and $d^2\Psi/dt^2 = -\omega_0^2\, \Psi_0 \sin \omega_0 t$)

Substituting into equation of motion, we find that the first and third terms on the left-hand-side cancel, leaving us with

$\Gamma\omega_0\Psi_0 \cos \omega_0 t = F_0 \cos \omega_0 t$, which is satisfied for all t if $\mathbf{\Psi_0 = F_0/(\Gamma\omega_0)}$

(2d) Velocity, $d\Psi/dt = \omega_0 \Psi_0 \cos \omega_0 t$

Power input = F $d\Psi/dt$

$= F_0 \Psi_0 \omega_0 \cos^2\omega_0 t = (F_0^2/\Gamma) \cos^2\omega_0 t = \Gamma \Psi_0^2 \omega_0^2 \cos^2\omega_0 t$

(Time average = $\frac{1}{2} F_0 \omega_0 \Psi_0 = \frac{1}{2}(F_0^2/\Gamma) = \frac{1}{2}\Gamma \Psi_0^2 \omega_0^2$)

The energy goes into heating the viscous liquid

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(3) Key points:

Hard tiled walls have very high impedance, because they move very little in response to a force (➔ Z = force/velocity is small)

Hence, sound waves in air are reflected efficiently when they hit the walls, which act as high impedance dashpots:

$R = (Z_{air} - Z_{wall}) / (Z_{air} + Z_{wall}) \sim -1$ ➔ reflected power ~ 100% and absorbed power is very small.

Similarly, water has a much higher impedance than air. Recall that for sound waves $Z = \sqrt{(\gamma p_0 \rho_0)}$. For water, $\gamma = d\ln p / d\ln \rho$ is very large, water being nearly incompressible, and the density ρ_0 is also much larger than for air.

Hence the water is very reflective as well: reflected power ~ 100% and transmitted power is very small.

Bottom line: sound waves created in the air by excited screaming children bounce around for a long time with very little energy loss ➔ it's very nosy in an indoor pool.

Underwater, however, the sounds are much fainter because of the poor **transmission** of the air/water interface.