Physics 171.106
Final Exam

May 16th, 2005

Answer all **seven** problems. Be sure that you pace yourself so that you have enough time to work on each problem. Note that the problems do not have equal weight. Partial credit will be given, so be sure to **show your work** as clearly as possible. Good luck!

<u>List of potentially useful formulae</u>

Coulomb's Law: $\vec{F}_2 = k\frac{q_1 q_2}{(\vec{r}_1 - \vec{r}_2)^2}\hat{r}_{21}$

Gauss's Law: $\oint \vec{E}\cdot d\vec{a} = 4\pi q_{inside} = 4\pi\int \rho dV$

$\vec{E}(\vec{r}_0) = \sum_{i=1}^{N}\frac{q_i}{(\vec{r}_i - \vec{r}_0)^2}\hat{r}_{i0}$

$U = \frac{1}{8\pi}\int \vec{E}^2 dV$

$\phi(\vec{r}_2) - \phi(\vec{r}_1) = -\int_{r_1}^{r_2}\vec{E}\cdot d\vec{s}$

$\vec{\nabla}\phi = -\vec{E}$ $\left(\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)$

Divergence Thm: $\oint \vec{E}\cdot d\vec{a} = \int(\vec{\nabla}\cdot\vec{E})dV$

Capacitance: $C = q/V$

$U = CV^2/2$

Force Law: $\vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}$

Ampere's Law: $\oint \vec{B}\cdot d\vec{s} = \frac{4\pi}{c}I_{enclosed}$

Stokes Thm: $\oint \vec{B}\cdot d\vec{s} = \int(\vec{\nabla}\times\vec{B})\cdot d\vec{a}$

$\varepsilon = \oint \vec{E}\cdot d\vec{s} = -\frac{1}{c}\frac{d\Phi}{dt}$

$$\vec{\nabla}\times\vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial Ex}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial Ey}{\partial x} - \frac{\partial Ex}{\partial y}\right)\hat{z}$$

Mutual inductance: $\varepsilon_{12} = -M_{21}\frac{dI_1}{dt}$ Self inductance: $\varepsilon_{11} = -L\frac{dI_1}{dt}$

Impedances: resistor $Z_R = R$ inductor $Z_L = i\omega L$ capacitor $Z_C = -i/\omega C$

Quality Factor: $Q = \omega$(energy stored)/(avg. power dissipated)

Maxwell's Equations:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$$

Wave Eq. (in vacuum): $\vec{\nabla}^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$

Poynting Vector: $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$

Dipole moment: $\vec{p} = \int d^3\vec{r}\rho(\vec{r})\vec{r}$

Polarization Density: $\vec{P} = N\vec{p} = \chi\vec{E}$ (χ = susceptibility)

Dielectric Constant: $\varepsilon = 1 + 4\pi\chi$

Ampere's Eq. in a dielectric: $\vec{\nabla}\times\vec{B} = \frac{\varepsilon}{c}\frac{\partial\vec{E}}{\partial t}$

Problem 1 (30 points)

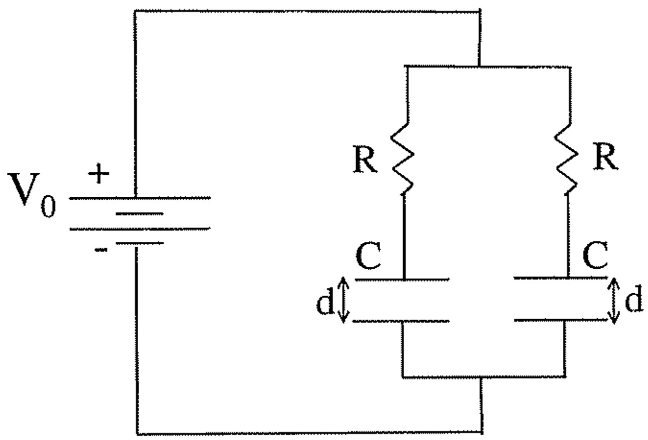
Consider the circuit below containing a battery with an emf of V_0, two resistors of equal value R and two parallel plate capacitors with equal vacuum capacitance C. The plates of the capacitors are separated by a distance d. Neglect fringe fields in the capacitors.

(a) Assuming that the circuit has been connected for a long time (so that transient effects can be ignored), determine the electric field **E** inside each of the capacitors.

(b) A dielectric material with dielectric constant $\varepsilon = 2$ is inserted into the capacitor on the right so that it completely fills the space between the plates. After any transient effects have died away, determine the electric field **E** inside each of the capacitors. Assume that the battery remains connected in the circuit as shown in the diagram.

(c) Again imagine that the dielectric material with $\varepsilon = 2$ is inserted into the capacitor. However, in this case the battery is removed from the circuit before the dielectric is inserted. Find the electric field **E** inside each capacitor after any transient effects have died away.

(d) For the case described in part (c), determine the total potential energy stored in the capacitors before and after the dielectric is inserted. Does the energy increase or decrease? If it increased, what supplied the energy; if it decreased, where did the energy go?

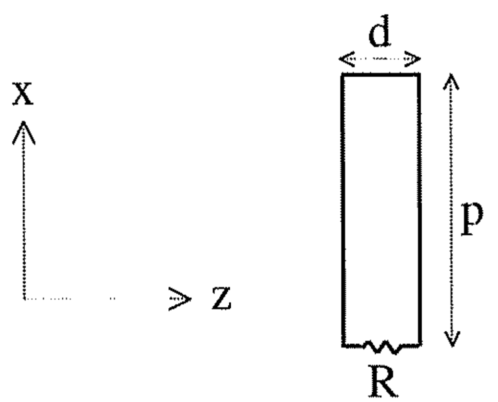


Problem 2 (20 points)

Consider a plane electromagnetic wave traveling in the negative-z direction. The wave's electric field has maximum magnitude of E_0 and is polarized along the x-direction. The wavelength of the wave is λ.

(a) Write down expressions for the electric and magnetic field vectors as a function of position and time in terms of the parameters E_0 and λ and fundamental constants.

(b) A wire antenna in the shape of a rectangular loop lies in the x-z plane. The loop has sides of length d and p as shown. The resistance of the loop is R. A current is induced in the loop as a consequence of Faraday induction. Assuming that $d \ll \lambda$, find the current as a function of time.



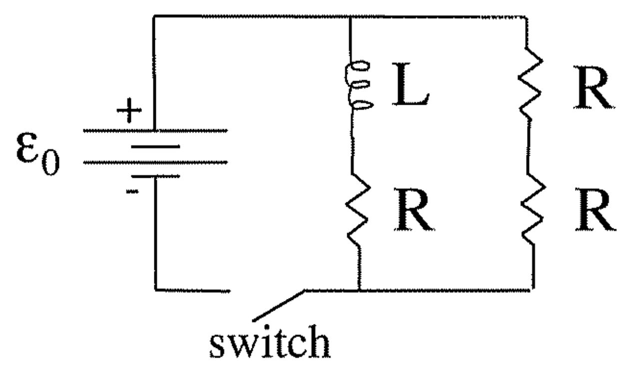
Problem 3 (30 points)

Consider the circuit in the picture below. All of the resistors have an equal value R. At time t = 0, the switch is closed.

(a) At a time very shortly after the switch is closed, what is the current flowing through the battery? At a very, very long time after the switch is closed, what is the current flowing through the battery?

(b) Derive an expression for the current flowing through the battery as a function of time after the switch is closed. Check that your expression has limits at short and long times that agree with your answers to part (a).

(c) Imagine that after the switch has been left closed for a long time, it is reopened at a time t_1. Find the energy stored in the inductor as a function of time after t_1.



Problem 4 (30 points)

Consider a cylindrical resistor of length L, radius a, and resistivity ρ. The cylinder is positioned with its axis along the z-axis. A current of magnitude I flows through the resistor in the positive-z direction. Assume that the current is distributed uniformly within the cylinder.

(a) Find the resistance R of the cylinder and the power P dissipated in it.

(b) Derive expressions for the electric field within the cylinder and for the magnetic field both inside and outside of the cylinder. Be sure to specify both the magnitude and direction of the fields. In determining the magnetic field, approximate the length of the cylinder as infinite.

(c) Calculate the Poynting vector at the surface of the cylinder.

(d) Based on your answer to part (c), determine the rate of electromagnetic energy flowing into or out of the resistor. How does your answer compare with your result for P from part (a)?

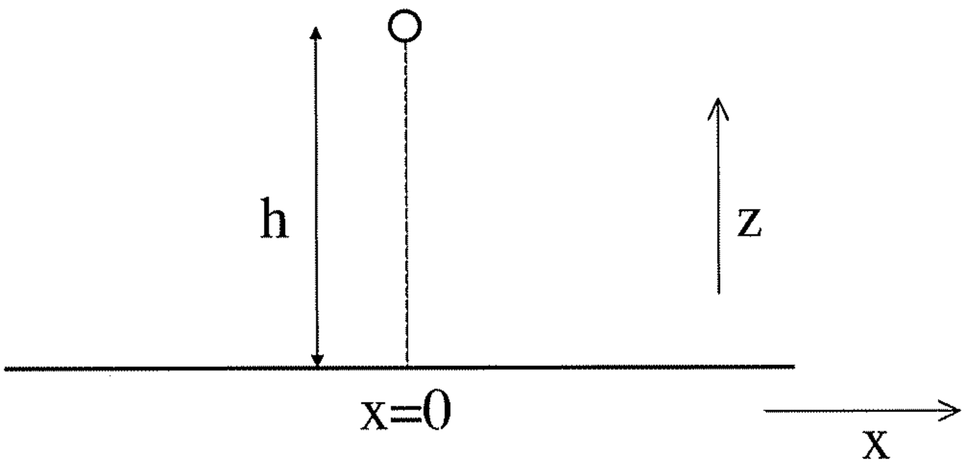
Problem 5 (30 points)

Consider an infinitely long line charge parallel to the y-axis at a height h above a grounded conducting plane as shown in the picture. "Grounded" means that charge if free to flow into the plane from a large reservoir of charge, and the potential at the plane is always zero. (The potential at infinity is also zero.) The line charge has a charge per unit length of λ.

(a) What is the electric field as a function of x for all points just above the plane?

(b) What is the surface charge density of the plane as a function of x?

(c) What is the electrostatic potential at the position, z=h/2, x=0?



Problem 6 (20 points)

A standing electromagnetic wave is confined between two parallel conducting planes positioned at z=0 and z=L. The electric field has the form:

$$\vec{E} = E_0 \sin\left(\frac{\pi z}{L}\right)\cos\left(\frac{\pi c t}{L}\right)\hat{x}$$

(a) What is the corresponding magnetic field?

(b) Find the surface current that flows in each of the conducting planes as a function of time.

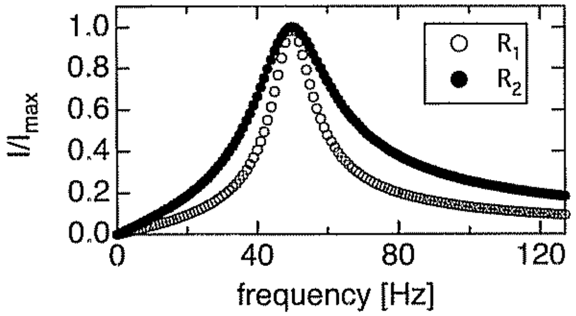
Problem 7 (40 points)

Each of the following **five** short problems is worth 8 points. In each case <u>be sure to explain your answer</u> in order to earn full credit.

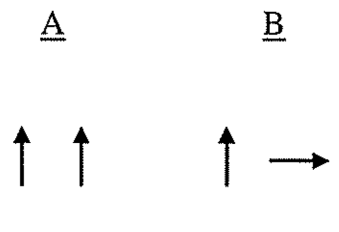
a) Consider the following three-dimensional vector field: $\vec{F} = x\hat{x} + y\hat{y}$.

Can this vector field be an electric field? Can it be a magnetic field?

b) A resistor R_1, an inductor, and a capacitor are wired in series with an AC power supply. The peak voltage from the power supply is held at a fixed value while the frequency is varied, and the current through circuit as a function of frequency is recorded. The resistor is replaced with a second resistor R_2 and the measurement is repeated. The graph below shows the current divided by the peak current for each case. Which resistor has a larger value?



c) The pictures below depict a pair of electric dipoles in different configurations. Order the configurations from lowest potential energy to highest potential energy



d) A traveling electromagnetic wave has a wavelength of λ in vacuum. The wave enters a medium with dielectric constant ε. What are the wavelength and frequency of the wave inside the medium?

e) A particle with charge q travels with a velocity v in the positive x-direction. The particle enters a region with a uniform magnetic field of magnitude B_0 oriented in the positive z-direction. The region also contains an electric field. The particle travels through the region with constant velocity. What is the magnitude and direction of the electric field?

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Problem #1

(a) $\boxed{E = V_0/d}$ for both capacitors

(b) As long as battery is connected, the voltage across the capacitors must be V_0

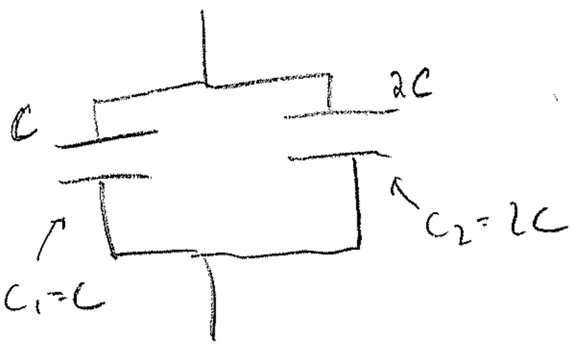
$\Rightarrow \boxed{E = V_0/d}$ for both capacitors

(c) Before dielectric is inserted charge on each capacitor is: $Q = CV_0 \Rightarrow$ total charge $= 2CV_0$

When dielectric the capacitor on the right will double in value. Charge will redistribute so that voltage across capacitors is the same

$$Q_{tot} = Q_1 + Q_2 = 2CV_0$$

$$V_1 = V_2 \Rightarrow \frac{Q_1}{C} = \frac{Q_2}{2C}$$



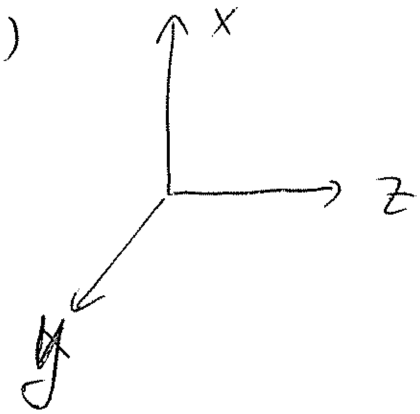
$$\Rightarrow Q_2 = 2Q \Rightarrow Q_1 = \frac{Q_{tot}}{3}$$

$$V_1 = \frac{2\cancel{C}V_0}{3\cancel{C}} = \frac{2}{3}V_0$$

$$\Rightarrow \boxed{E = \frac{2}{3}\frac{V_0}{d}}$$

Problem #2

(a)



If we take

$$\vec{E} = \hat{x} E_0 \cos\left[\frac{2\pi}{\lambda}(ct+z)\right]$$

Then

$$\vec{B} = -\hat{y} B_0 \cos\left[\frac{2\pi}{\lambda}(ct+z)\right]$$

where $B_0 = E_0$

(b) The electromotive force is

$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{pd}{c}\frac{dB}{dt} \qquad (d \ll \lambda)$$

$$\Rightarrow \mathcal{E} = -\frac{pdB_0}{c} \cdot \frac{2\pi c}{\lambda} \cdot \cos\left[\frac{2\pi}{\lambda}(ct+z)\right]$$

$$\Rightarrow I = \frac{\mathcal{E}}{R} = -\frac{2\pi p d B_0}{\lambda R} \cos\left[\frac{2\pi}{\lambda}(ct+z)\right]$$

$$U_i = \frac{1}{2}CV_0^2 + \frac{1}{2}CV_0^2 = \boxed{CV_0^2}$$

$$U_f = \frac{1}{2}C\left(\frac{2}{3}V_0\right)^2 + \frac{1}{2}(2C)\left(\frac{2}{3}V_0\right)^2 =$$

$$U_f = \frac{2}{9}CV_0^2 + \frac{4}{9}CV_0^2 = \boxed{\frac{2}{3}CV_0^2 < U_i}$$

Energy <u>decrease</u>. It was dissipated in the resistors as charge flowed from resistor on the right to resistor on the left.

Problem #3

(a) At a time just after the switch is closed, we have (No current through the inductor)

$$I_B = \mathcal{E}_0 / 2R$$

At a very, very long time after the switch is closed, we have ~~(inductor equivalent to a wire)~~

$$I_B = \mathcal{E}_0 / \left(\frac{2R}{3}\right) = \frac{3\mathcal{E}_0}{2R}$$

(b) Denote the current through the inductor by $I(t)$. Take the time when the switch is closed as 0.

Then

$$IR = \mathcal{E}_0 + \left(-L\frac{dI}{dt}\right)$$

$$\Rightarrow \frac{dI}{dt} + \frac{R}{L}I - \frac{\mathcal{E}_0}{L} = 0$$

$$\Rightarrow I(t) = C_1 e^{-\frac{Rt}{L}} + C_2$$

$$\Rightarrow \frac{R}{L}\cdot C_2 - \frac{\mathcal{E}_0}{L} = 0 \Rightarrow C_2 = \frac{\mathcal{E}_0}{R}$$

And when $t=0$, $I(0)=0$, $\Rightarrow C_1 = -C_2 = -\frac{\mathcal{E}_0}{R}$

$$\Rightarrow I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-Rt/L}\right)$$

The current I_B through the battery as a function of time is

$$I_B(t) = I(t) + \frac{\mathcal{E}_0}{2R} = \frac{3\mathcal{E}_0}{2R} - \frac{\mathcal{E}_0}{R}e^{-\frac{Rt}{L}}$$

When $t=0$, we get

$$I_B(0) = 3\mathcal{E}_0/2R - \mathcal{E}_0/R = \mathcal{E}_0/2R$$

When $t=\infty$, we get

$$I_B(\infty) = 3\mathcal{E}_0/2R$$

which are the same as our answers to part (a).

(c) At time t_1, the current through the inductor is

$$I(t_1) = \frac{\mathcal{E}_0}{R}\left(1-e^{-Rt_1/L}\right)$$

And at this time, the switch is reopened, so

$$I(t)\cdot 3R = -L\frac{dI(t)}{dt} \quad (t > t_1)$$

$$\Rightarrow I(t) = I(t_1)\, e^{-\frac{3R(t-t_1)}{L}}$$

$$= \frac{\mathcal{E}_0}{R}\left(1-e^{-\frac{Rt_1}{L}}\right)e^{-\frac{3R(t-t_1)}{L}}$$

The energy stored in the inductor is

$$U = \frac{1}{2}LI^2(t) = \frac{\mathcal{E}_0^2 L}{2R^2}\left(1-e^{-\frac{Rt_1}{L}}\right)^2 e^{-\frac{6R(t-t_1)}{L}}$$

where $t > t_1$.

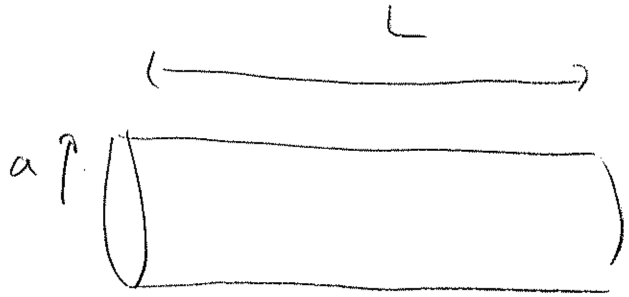
Since t_1 is very long, we can get ($e^{-\frac{Rt_1}{L}} \approx 0$)

$$I(t) \simeq \frac{\mathcal{E}_0}{R}e^{-\frac{3R(t-t_1)}{L}} = \frac{\mathcal{E}_0}{R}e^{-\frac{3R\Delta t}{L}}$$

$$U(t) \simeq \frac{\mathcal{E}_0^2 L}{2R^2}e^{-\frac{6R(t-t_1)}{L}} = \frac{\mathcal{E}_0^2 L}{2R^2}e^{-\frac{6R\Delta t}{L}}$$

($\Delta t = t - t_1$)

Problem # 4



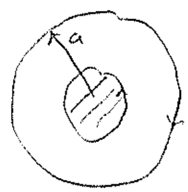
(a) $\boxed{R = \rho \frac{L}{\pi a^2}}$ $\quad P = I^2 R = \boxed{\frac{I^2 \rho L}{\pi a^2}}$

(b) Voltage across cylinder $= IR$

$$\bar{E} = V/L = IR/L = \boxed{\frac{I\rho}{\pi a^2}\hat{z}}$$

For magnetic field, use Ampere's Law:

$$\oint B \cdot d\ell = \frac{4\pi}{c} I_{enclosed}$$



$$J = \frac{I}{\pi a^2}$$

$r < a$

$$B \cdot 2\pi r = \frac{4\pi}{c}(\pi r^2 J)$$

$$B \cdot 2\pi r = \frac{4\pi}{c}\pi r^2 \frac{I}{\pi a^2}$$

$$\bar{B} = \frac{4\pi^2 r^2 I}{2\pi^2 r c a^2} = \boxed{\frac{-2I}{ca^2} r \hat{\theta}}$$

$r \geq a$:

$$B \cdot 2\pi r = \frac{4\pi}{c} I$$

$$\boxed{\bar{B} = \frac{-2I}{cr}\hat{\theta}}$$

(c) $\bar{S} = \frac{c}{4\pi} \bar{E} \times \bar{B}$

@ surface $\quad \bar{E} = \frac{I\rho}{\pi a^2} \hat{z}$

$$\bar{B} = -\frac{2I}{ca} \hat{\theta}$$

$$\Rightarrow \quad \bar{S} = \frac{\cancel{c}}{4\pi}\left(\frac{I\rho}{\pi a^2}\right)\left(\frac{2I}{\cancel{c}a}\right)(-\hat{r})$$

↖ inward

$$\boxed{\bar{S} = -\frac{\rho I^2}{2\pi^2 a^3} \hat{r}}$$

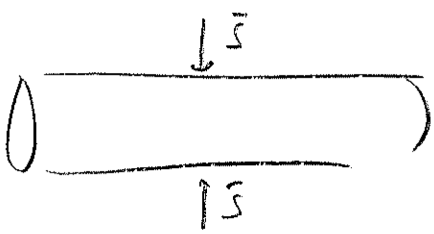
(d) Poynting vector is rate of energy flow (power) per unit area

$$\Rightarrow \quad P = \bar{S} \cdot \bar{A}$$

↖ surface of cylinder

$$= S \cdot (2\pi a L)$$

$$\Rightarrow \quad P = \frac{\rho I^2}{2\pi^2 a^3} 2\pi a L$$



$$= \boxed{\frac{I^2 \rho L}{\pi a^2}}$$

same as part (a).

Problem # 6

(a) $$\bar{\nabla}\times\bar{E} = \frac{-1}{c}\frac{\partial\bar{B}}{\partial t}$$

$$\Rightarrow \quad \frac{\partial E_x}{\partial z}\hat{y} = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t}$$

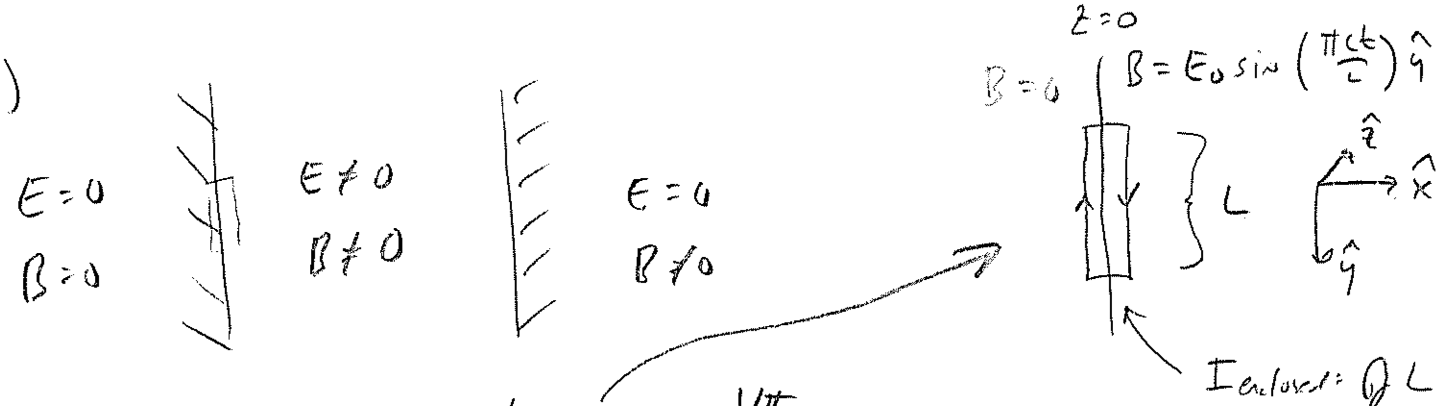
$$E_0\frac{\pi}{L}\cos\left(\frac{\pi z}{L}\right)\cos\left(\frac{\pi c t}{L}\right)\hat{y} = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$\cancel{-}\,E_0\cancel{\frac{\pi}{L}}\left(\cancel{\frac{L}{\pi c}}\right)\cos\left(\frac{\pi z}{L}\right)\sin\left(\frac{\pi c t}{L}\right)\hat{y} = \cancel{-\frac{1}{c}}\,\bar{B}$$

(NOTE: There can also be a time independent contribution to $\bar{B}$ that we can ignore.)

$$\Rightarrow \quad \boxed{\bar{B} = E_0\cos\left(\frac{\pi z}{L}\right)\sin\left(\frac{\pi c t}{L}\right)\hat{y}}$$

(b)

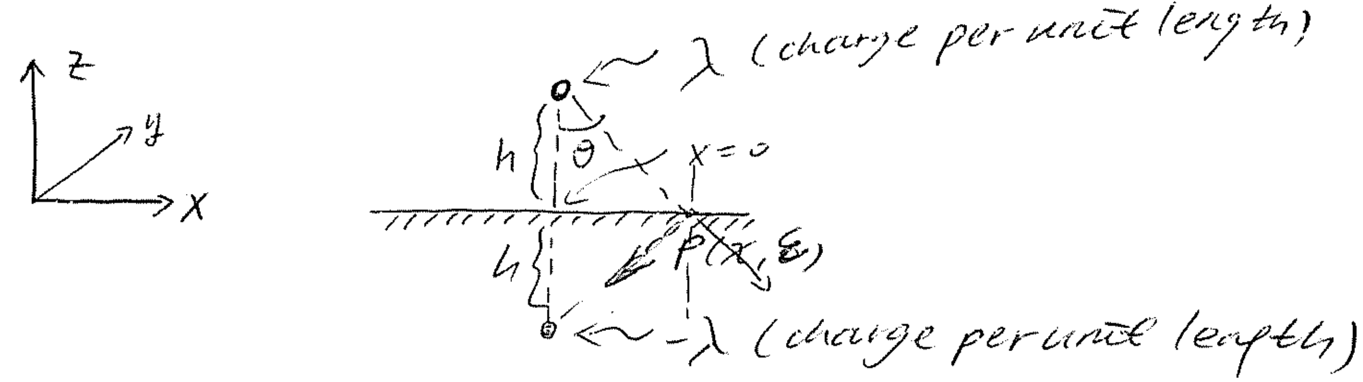


By Ampere's Law $$\oint B\cdot d\ell = \frac{4\pi}{c}I_{enclosed}$$

$$B\cdot \cancel{L} = \frac{4\pi}{c}J\cancel{L}$$

$$\boxed{J = \frac{c}{4\pi}E_0\sin\left(\frac{\pi c t}{L}\right)}$$

Problem #5



The image line charge is just at a depth h below the conducting plane and has a charge per unit length of $-\lambda$.

(a) At a point $p(x, \varepsilon)$, $\varepsilon \to 0$, we have

$$\vec{E} = -\hat{z}\left[\frac{2\lambda}{\sqrt{h^2+x^2}} \cdot \underbrace{\frac{h}{\sqrt{h^2+x^2}}}_{=\cos\theta} \times 2\right]$$

$$= -\frac{4\lambda h}{(x^2+h^2)}\hat{z}$$

Here we use at a point with a distance r from an infinitely long line charge of linear charge density λ, the electric field is along the radius and the magnitude is $\frac{2\lambda}{r}$

(b) By $\oint_S \vec{E}\cdot d\vec{a} = 4\pi q_{\text{enclosed by } S}$, we know at the surface of a conducting plane, the electric field is perpendicular to that plane, and

$$\Delta E = 4\pi\sigma$$

ΔE is the jump of the electric field across the surface

$\Rightarrow$ The surface charge density of the plane is

$$\sigma = \frac{\Delta E}{4\pi} = -\frac{\lambda h}{\pi (x^2+h^2)}$$

(Inside the conductor $E = 0$. Outside $E = -\frac{4\lambda h}{(x^2+h^2)}$)

(C) At $x=0$, the electric field is

~~[illegible]~~

$$E(0,z) = -\frac{2\lambda}{-z+h} - \frac{2\lambda}{z+h} \quad (z>0)$$

The electrostatic potential at the position $z = h/2$, $x=0$ is just

$$\phi = -\int_0^{h/2} E(0,z)\, dz$$

$$= +\int_0^{h/2} \left(\frac{2\lambda}{h-z} + \frac{2\lambda}{h+z}\right) dz$$

$$= 2\lambda \ln\frac{3h}{2} - \ln\frac{h}{2}$$

$$= 2\lambda \ln 3$$ ~~(electrostatics)~~ statvolts.

Actually, $\phi(0,z) = 2\lambda\left[\ln\left(\frac{h+z}{h-z}\right)\right]$

for $0 \leq z < h$

Problem # 7

(a) $F = x\hat{x} + y\hat{y}$

$\bar{\nabla} \cdot \bar{F} = 2 \neq 0$

$\bar{\nabla} \times \bar{F} = 0$

$\Longrightarrow$ can be $\bar{E}$ field; can **not** be $\bar{B}$ field

(b) R_2 is larger since peak is broader.

width $\sim \frac{1}{Q}$ ← quality factor

(c) C lowest

B middle

A highest

(d) wavelength $= \lambda / \varepsilon$

frequency $= \frac{2\pi c}{\lambda}$ (same as in vacuum)

(e) $\bar{F} = q\bar{E} + \frac{q}{c}\,\bar{v} \times \bar{B} = 0$ $\qquad \bar{V} = v\hat{x}$, $\bar{B} = B_0\hat{z}$

$$\Longrightarrow \boxed{\bar{E} = \frac{v}{c} B_0 \hat{y}}$$