Physics 171.106
Midterm Exam 2

April 13th, 2005

Answer all **four** problems. Be sure that you pace yourself so that you have enough time to work on each problem. Note that the problems have equal weight. Partial credit will be given, so be sure to **show your work** as clearly as possible. Good luck!

List of potentially useful formulae

Ohm's Law: $\vec{J} = \sigma\vec{E}$ (a.k.a. $V = IR$)

Biot-Savart Law: $d\vec{B} = I\dfrac{d\vec{l} \times \hat{r}}{cr^2}$

Force Law: $\vec{F} = q\vec{E} + \dfrac{q}{c}\vec{v} \times \vec{B}$

Ampere's Law: $\oint \vec{B} \cdot d\vec{s} = \dfrac{4\pi}{c} I_{enclosed}$

Stokes Thm: $\oint \vec{B} \cdot d\vec{s} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a}$

Faraday's Law: $\vec{\nabla} \times \vec{E} = -\dfrac{1}{c}\dfrac{\partial \vec{B}}{\partial t}$

Capacitance: $C = q/V$

$$\varepsilon = \oint \vec{E} \cdot d\vec{s} = -\frac{1}{c}\frac{d\Phi}{dt}$$

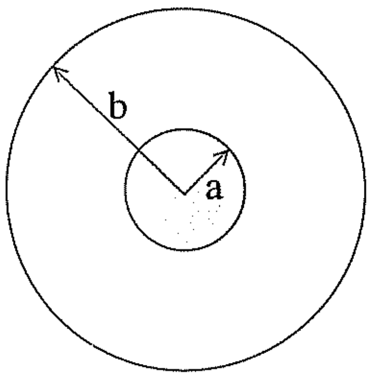
$$\vec{\nabla} \times \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial Ex}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial Ey}{\partial x} - \frac{\partial Ex}{\partial y}\right)\hat{z}$$

Mutual inductance: $\varepsilon_{12} = -M_{21}\dfrac{dI_1}{dt}$

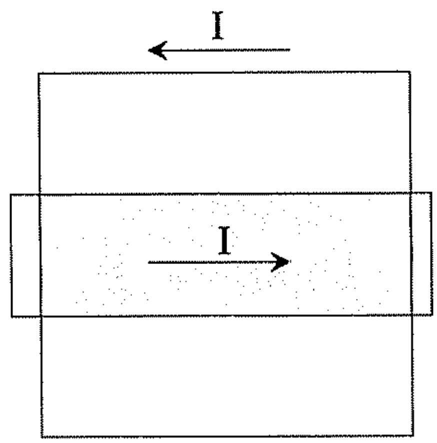
Self inductance: $\varepsilon_{11} = -L\dfrac{dI_1}{dt}$

Problem 1 (25 points)

Consider an infinitely long coaxial cable, as shown in the figure below. The inner conductor of the cable is a solid cylinder of radius a. The outer conductor is a thin cylindrical shell of radius b. A current I flows through the inner conductor and returns in the opposite direction through the outer conductor as shown. The current in the inner conductor is distributed with a uniform current density. Similarly, the current in the outer conductor is distributed as a uniform "sheet" density on the shell. What is the magnitude of the magnetic field in the three regions $r < a$, $a < r < b$, and $r > b$?



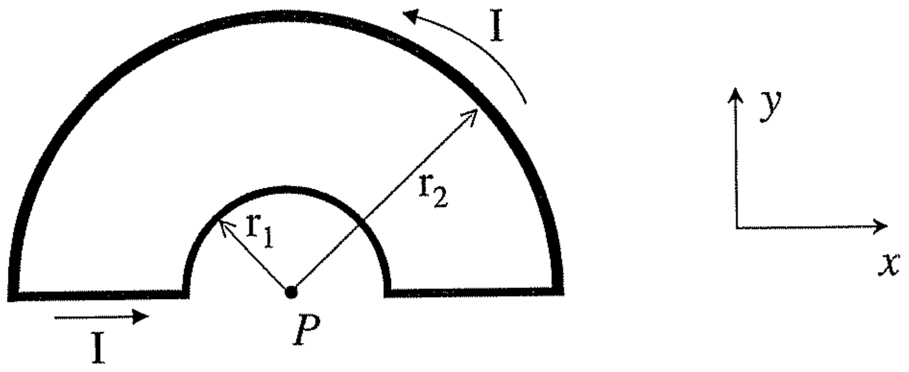
face view



side view

Problem 2 (25 points)

Consider a wire loop bent into the shape of concentric semicircles, as shown in the picture below. The inner and outer radii of the semicircles are r_1 and r_2, respectively. A current I flows in the loop in the direction shown by the arrows. Determine the magnitude and direction of the magnetic field at the point *P* at the center of the semicircles.



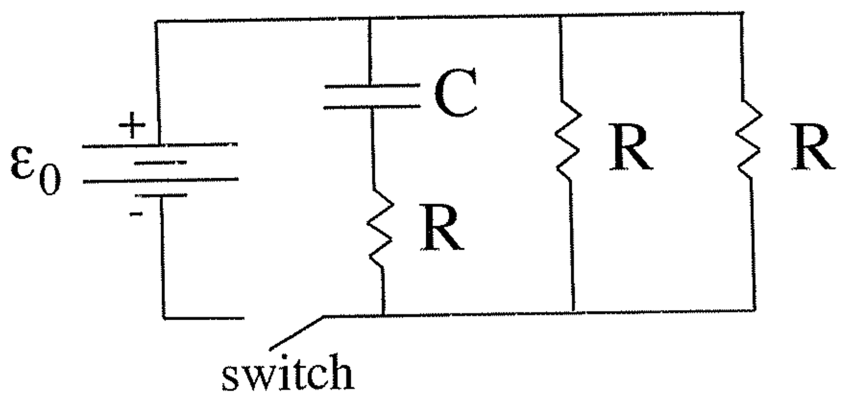
Problem 3 (25 points)

Consider the circuit in the picture below. All of the resistors have an equal value R. Initially, the capacitor C is uncharged. At time $t = 0$, the switch is closed.

(a) In the instant after the switch is closed, what is the current flowing through the battery?

(b) At a very, very long time after the switch is closed, what is the current flowing through the battery? How much charge is on the capacitor at this late time?

(c) Imagine that after the switch has been left closed for a long time, it is reopened at a time t_1. Find the charge on the capacitor as a function of time after t_1.



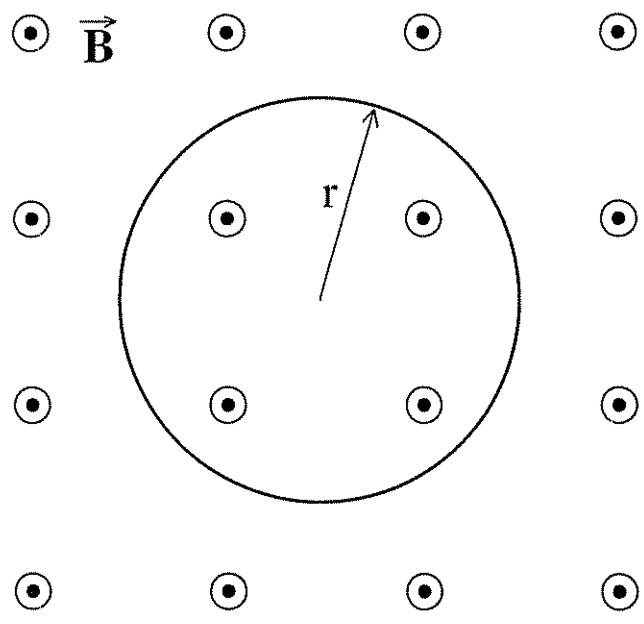
Problem 4 (25 points)

Consider a circular loop of wire in a uniform magnetic field of size B oriented perpendicular to the loop, as shown below. (The magnetic field in the picture is coming out of the page.) The loop is made of a special material such that the radius of the loop increases linearly with time as $r = r_0(1+\alpha t)$, where t is time and α is a numerical constant. Because the loop grows thinner as it expands, its resistance increases linearly with time as well: $R = R_0(1+\alpha t)$. Note that the numerical constant α is the same in both expressions.

(a) Determine the electromotive force that is generated in the loop as a function of time. How would your answer change if instead the field were oriented in the plane of the loop (i.e., in the plane of the page)?

(b) Does the current that is induced in the loop flow clockwise or counter-clockwise, as viewed in the picture below? Why?

(c) How much power is dissipated in the loop as a function of time?



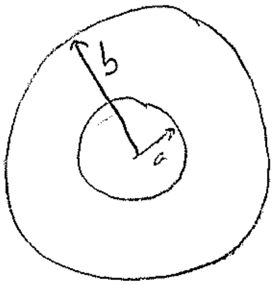
171.106 Midterm II 4/13/05

Solutions

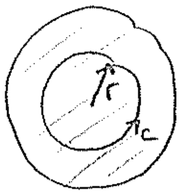
(1)

By symmetry, the magnetic field points in $\hat{\theta}$ and has a magnitude that depends only on r

$\Rightarrow$ Use Ampere's Law: $\oint_c \bar{B}\cdot d\bar{s} = \frac{4\pi}{c} I_{enclosed}$



$\underline{r<a}$ $J = \frac{I}{\pi a^2}$ $B\cdot 2\pi r = \frac{4\pi}{c} I_{enc} = \frac{4\pi}{c} J \pi r^2$

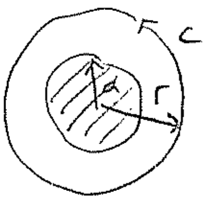


$$B = \frac{4\pi I \cancel{\pi} r^{\cancel{2}}}{2\cancel{\pi}\cancel{r}\,\cancel{\pi} a^2 c}$$

$$\boxed{B = \frac{2Ir}{a^2 c}}$$

$\underline{a<r<b}$ $B\cdot 2\pi r = \frac{4\pi}{c} I$

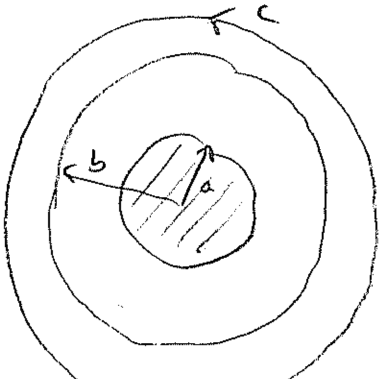
$$\Rightarrow \boxed{B = \frac{2I}{cr}}$$



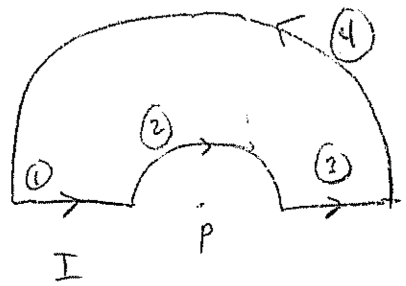
$\underline{r>b}$

$I_{enclosed} = I - I = 0$

$$\Rightarrow \boxed{B = 0}$$



(2)



$$d\bar{B} = I \frac{d\bar{\ell} \times \hat{r}}{cr^2}$$

For segments ① & ③, $d\bar{\ell}_{\parallel} \hat{r} \Rightarrow d\bar{\ell} \times \hat{r} = 0$

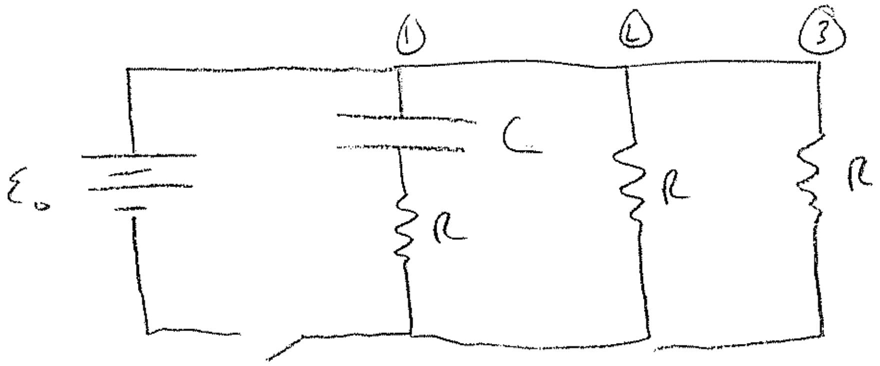
For segments ② & ④, $d\bar{\ell}_{\perp} \hat{r} \Rightarrow d\bar{\ell} \times \hat{r} = r\,d\theta$

$$\Rightarrow \bar{B} = \left(\int_0^{\pi} \frac{I r_1 \, d\theta}{c r_1^2} - \int_0^{\pi} \frac{I r_2 \, d\theta}{c r_2^2} \right) (-\hat{z})$$

$$\bar{B} = \frac{I\pi}{c} \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \hat{z}$$

(NOTE: direction determined from right hand rule)

(3)



(a) right after the switch is closed, the charge on the capacitor is negligably small, so can ignore the voltage drop across capacitor. In this case branch 1 is effectively just like branches 2 & 3.

$$\Rightarrow \quad \frac{1}{R_{tot}} = \frac{1}{R} + \frac{1}{R} + \frac{1}{R} = \frac{3}{R}$$

$$I = \frac{\mathcal{E}_0}{R_{tot}} = \boxed{\frac{3\mathcal{E}_0}{R}}$$

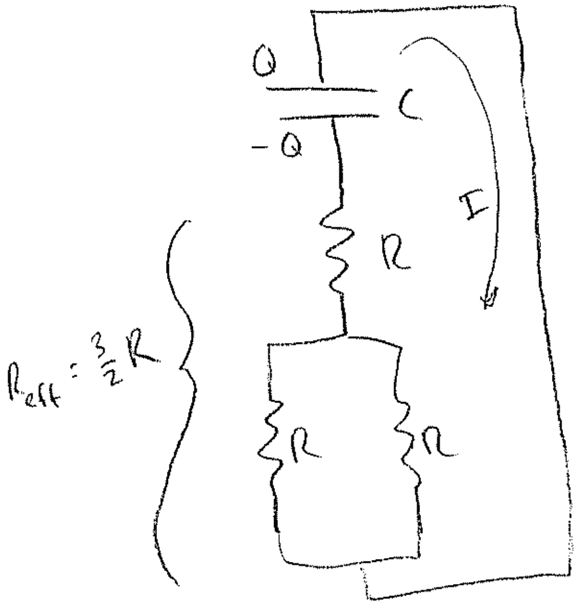
(b) After a very long time, charge will have collected on the capacitor to the point that the voltage drop across the capacitor is $\mathcal{E}_0$, and no more current passes through branch 1. In this case

$$\frac{1}{R_{tot}} = \frac{1}{R} + \frac{1}{R} = \frac{2}{R}$$

$$\Rightarrow \quad I = \frac{\mathcal{E}_0}{R_{tot}} = \boxed{\frac{2\mathcal{E}_0}{R}}$$

& the charge on the capacitor is $\boxed{Q = C\mathcal{E}_0}$

(c) When the switch is opened, the circuit looks like:



Kirchoff's Laws:

$$\frac{Q}{C} - \frac{3}{2}RI = 0 \qquad I = -\frac{dQ}{dt}$$

$$Q + \frac{3}{2}RC\frac{dQ}{dt} = 0$$

$$\Rightarrow \quad Q = Q_0 e^{-(t-t_1)/\frac{3}{2}RC}$$

where Q_0 is the initial charge, which is $C\mathcal{E}_0$

$$\Rightarrow \quad \boxed{Q = C\mathcal{E}_0 e^{-(t-t_1)/\frac{3}{2}RC}}$$

(4) (a) Flux $\Phi = \int B \cdot da = B\pi r^2 = B\pi r_0^2 (1+\alpha t)^2$

$$\mathcal{E}mf = -\frac{1}{c}\frac{d\Phi}{dt} = \boxed{B\pi r_0^2\, 2\alpha\, (1+\alpha t)/c}$$

If $\bar{B}$ were in plane of loop then $\Phi = 0 \implies \boxed{\mathcal{E}MF = 0}$

(b) Since the flux is increasing with time, by Lenz's law the current generated by the EMF will fight this increase by creating an induced field into the page. By the right hand rule, a $\boxed{\text{clockwise}}$ current produces such a field.

(c) $$P = \frac{V^2}{R} = \frac{(B\pi r_0^2 2\alpha)^2 (1+\alpha t)^{\cancel{2}}}{c^2 R_0 \cancel{(1+\alpha t)}}$$

$$= \boxed{\frac{4\pi^2 B^2 \alpha^2 r_0^4}{R_0 c^2} (1+\alpha t)}$$