Quantum Mechanics 171.304

Summary of 2006 Spring Semester

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Two body Problem

$$\hat{H} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(|\mathbf{r}_1 - \mathbf{r}_2|)$$

Center of mass frame + spherical symmetry →

$$\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\hbar^2\ell(\ell+1)}{2\mu r^2} + V(r)\right)u(r) = Eu(r) \qquad \langle r\theta\varphi | Elm\rangle = \frac{u(r)}{r}Y_{lm}(\theta\varphi)$$

Because $\hat{\mathbf{L}}$ is a differential operator in position space: $\ell = 0,1,2,\ldots$

$$Y_{\ell|m|}(\theta\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}e^{im\varphi}P_\ell^m(\cos\theta) \qquad Y_{\ell-|m|}(\theta\varphi) = (-1)^m\left[Y_{\ell|m|}(\theta\varphi)\right]^*$$

Are position space eigenstates of the angular momentum operator

Infinite spherical well: $E_{n\ell} = \frac{\hbar^2}{2\mu a^2}\beta_{n\ell}^2 \qquad \psi_{n\ell m}(r\theta\varphi) = A_{n\ell}\, j_\ell(\beta_{n\ell} r/a)Y_{\ell m}(\theta\varphi)$

Finite spherical well bound states only for: $V_0 > \frac{\pi^2\hbar^2}{8\mu a^2}$

Spherical harmonic oscillator is three 1D oscillators: $E_{n\ell} = \hbar\omega\left(2n + \ell + \tfrac{3}{2}\right)$

:

Perturbation Theory

Hamiltonian has a main “simple” part and a weak “complicated” part

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

Non-degenerate:

$$E_n \approx E_n^{(0)} + \left\langle \varphi_n^{(0)} \middle| \hat{H}_1 \middle| \varphi_n^{(0)} \right\rangle + \sum_{k \neq n} \frac{\left| \left\langle \varphi_k^{(0)} \middle| \hat{H}_1 \middle| \varphi_n^{(0)} \right\rangle \right|^2}{E_n^{(0)} - E_k^{(0)}} + \ldots$$

$$\left| \psi_n \right\rangle \approx \left| \varphi_n^{(0)} \right\rangle + \sum_{k \neq n} \left| \varphi_k^{(0)} \right\rangle \frac{\left\langle \varphi_k^{(0)} \middle| \hat{H}_1 \middle| \varphi_n^{(0)} \right\rangle}{E_n^{(0)} - E_k^{(0)}} + \ldots$$

Degenerate: Diagonalize perturbing operator in degenerate subspace

$$\left(H_1\right)_{nm} c_m = E^{(1)} c_n$$

At each stage of perturbation diagonalize only degenerate subspace

Hydrogen Atom

Non-relativistic spin less bound state problem:

$$E_n = -\frac{\mu c^2 \alpha^2}{2n^2} = -\frac{\hbar^2}{2\mu}\left(\frac{1}{na_0}\right)^2 \qquad \alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \qquad a_0 = \frac{\hbar}{\mu c \alpha}$$

$$\langle r\theta\varphi | n\ell m\rangle = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n((n+\ell)!)^3}} e^{-r/na_0}\left(\frac{2r}{na_0}\right)^\ell L_{n-\ell-1}^{2\ell+1}(2r/na_0) Y_{\ell m}(\theta\varphi)$$

Relativistic correction, spin-orbit coupling and Darwin term

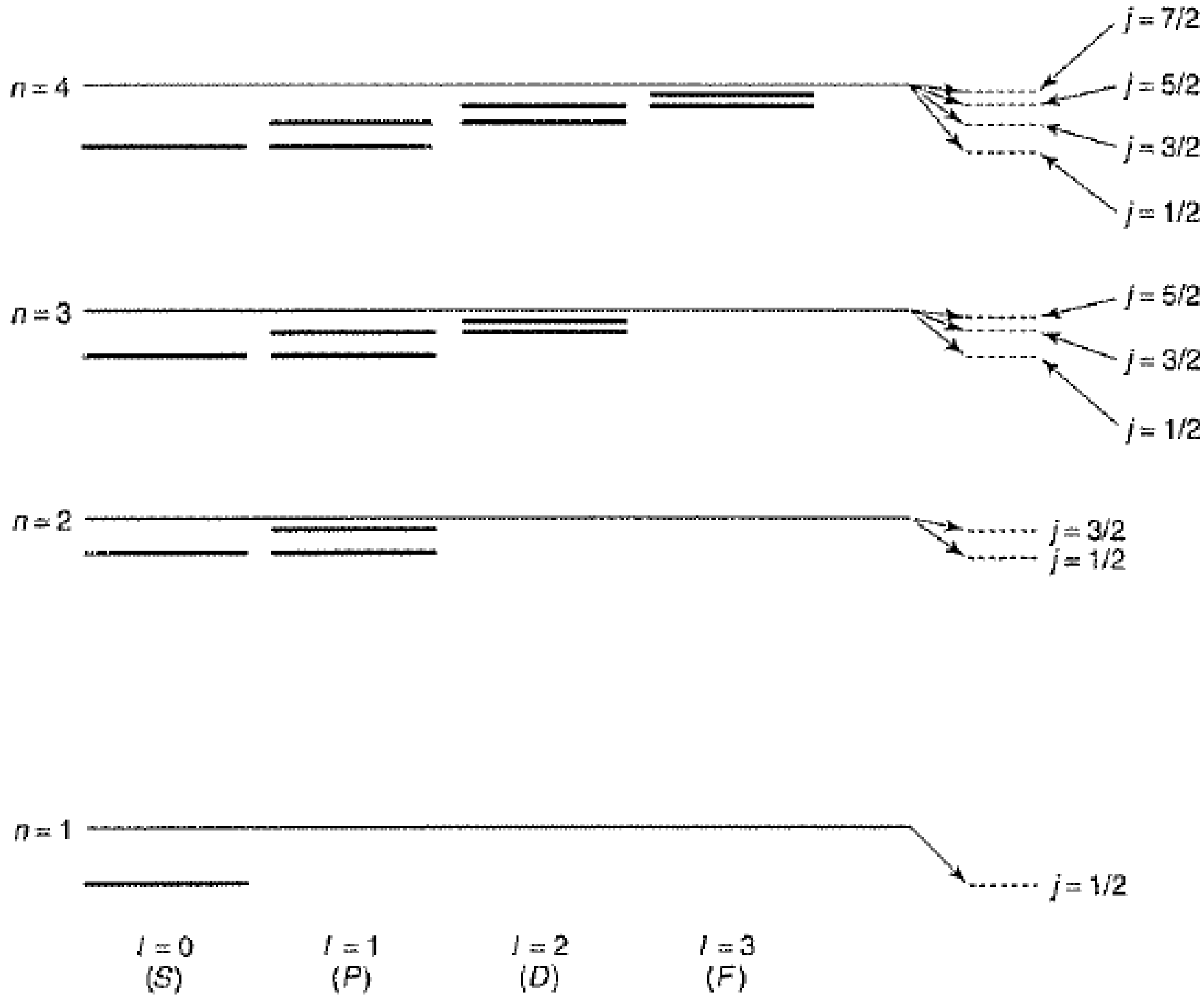
$$E_{nj} = E_n\left(1 + \left(\frac{\alpha}{n}\right)^2\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right)$$

Hyperfine interaction is electron-proton dipole interaction

$$\Delta E = \frac{4 g_p \hbar^4}{3 m_p m_e c^2 a_0^4} \qquad g_p = 5.59$$

Famous λ=21 cm line used to probe
Speed of cold gas in astrophysics

Hydrogen Atom Level Scheme



Identical particles

Bosons: $\psi(\mathbf{r}_2, \mathbf{r}_1) = \psi(\mathbf{r}_1, \mathbf{r}_2)$ $s = 1,2,3,..$

Fermions: $\psi(\mathbf{r}_2, \mathbf{r}_1) = -\psi(\mathbf{r}_1, \mathbf{r}_2)$ $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...$

Boson are closer

$$\left\langle (x_1 - x_2)^2 \right\rangle = \left\langle x^2 \right\rangle_n + \left\langle x^2 \right\rangle_m - 2\langle x \rangle_n \langle x \rangle_m \mp 2\left|\langle x \rangle_{nm}\right|^2$$

Fermions further apart

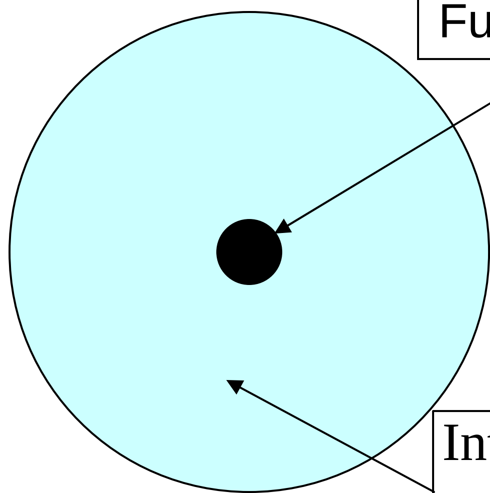
Variational technique

For any normalized wave function

$$\langle \psi | \hat{H} | \psi \rangle \geq E_0$$

Use these results to obtain estimate for ground state energy of helium and to consider the level scheme for multi-electron atoms.

Partial Wave analysis of scattering



Full problem

Radiation zone

$$\psi(\mathbf{r}) = A\left(e^{ikz} + f(\theta\varphi)\frac{e^{ikr}}{r} \right)$$

Intermediate zone

$$\psi(r\theta) = A\sum_{\ell} i^{\ell}(2\ell+1)\left(j_{\ell}(kr) + ika_{\ell}h_{\ell}(kr)\right)P_{\ell}(\cos\theta)$$

$$\frac{d\sigma}{d\Omega} = \frac{\text{particles into solid angle d}\Omega\text{ per time unit}}{\text{incident particles per time and area unit}} = |f|^2$$

$$\sigma = 4\pi\sum_{\ell}(2\ell+1)|a_{\ell}|^2$$

Phase shift analysis of scattering

In analogy with phase shifts in 1D use this expression in intermediate zone:

$$\psi(r\theta) = A\sum_{\ell} \frac{2\ell+1}{2ikr}\left(e^{i(kr+2\delta_\ell)} - (-1)^\ell e^{-ikr}\right)P_\ell(\cos\theta)$$

The connection to partial wave amplitudes and scattering cross section

$$a_\ell = \frac{1}{k}e^{i\delta_\ell}\sin\delta_\ell$$

$$\sigma = 4\pi\sum_{\ell}(2\ell+1)\frac{1}{k^2}\sin^2\delta_\ell = \frac{4\pi}{k}\mathrm{Im}\{f(0)\}$$ Optical theorem

Born Approximation

$$f(\theta\varphi) = -\frac{m}{2\pi\hbar^2}\int d^3\mathbf{r}' V(\mathbf{r}')e^{i\mathbf{q}\cdot\mathbf{r}'} \qquad \mathbf{q} = \mathbf{k}_i - \mathbf{k}_f$$

Is good approximation when scattering sufficiently weak

Quantum Statistical Mechanics

$$n(\varepsilon) = \begin{cases} e^{-\beta(\varepsilon-\mu)} & \text{Distinguishable “classical” particles} \\ \dfrac{1}{e^{\beta(\varepsilon-\mu)}-1} & \text{Bosons with symmetric } \psi \\ \dfrac{1}{e^{\beta(\varepsilon-\mu)}+1} & \text{Fermions with anti-symmetric } \psi \end{cases}$$

Use “super-Mario” boundary conditions to count k-states in volume V

⇨ One k-state occupies volume $\dfrac{(2\pi)^3}{V}$ of k-space

$$\langle F(\varepsilon)\rangle = \int F(\varepsilon) g(\varepsilon) n(\varepsilon) d\varepsilon$$

$g(\varepsilon)$ Is energy dependent density of available states

Chemical potential is determined by requiring $N = \int g(\varepsilon) n(\varepsilon) d\varepsilon$

Time dependent perturbations

Relationship between different “pictures” of quantum mechanics

Picture	States related	Operators related	States versus t	Operators versus t
Schrödinger	$\lvert\psi_S(t)\rangle$	$\hat{O}_S$	$i\hbar\frac{d}{dt}\lvert\psi_S(t)\rangle = \hat{H}\lvert\psi_S(t)\rangle$	t-independent *
Interaction	$\lvert\psi_I(t)\rangle = \exp(i\hat{H}_0 t/\hbar)\lvert\psi_S(t)\rangle$	$\hat{O}_I(t) = \exp(i\hat{H}_0 t/\hbar)\hat{O}_S \exp(-i\hat{H}_0 t/\hbar)$	$i\hbar\frac{d}{dt}\lvert\psi_I(t)\rangle = \hat{H}_{1I}(t)\lvert\psi_S(t)\rangle$	$\frac{d\hat{O}_I}{dt} = \frac{i}{\hbar}[\hat{H}_0, \hat{O}_I]$
Heisenberg	$\lvert\psi_H(t)\rangle = \exp(i\hat{H}t/\hbar)\lvert\psi_S(t)\rangle$	$\hat{O}_H(t) = \exp(i\hat{H}t/\hbar)\hat{O}_S \exp(-i\hat{H}t/\hbar)$	t-independent	$\frac{d\hat{O}_H}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{O}_H]$

* An operator in the Schrödinger picture can have explicit t-dependence as for the time dependent perturbation below.

$$\hat{H} = \hat{H}_0 + \hat{H}_1(t)$$

$$\lvert\psi_S(0)\rangle = \left\lvert E_k^{(0)}\right\rangle \qquad \lvert\psi_S(t)\rangle = \sum_n c_n(t)\exp\left(-iE_n^{(0)}t/\hbar\right)\left\lvert E_n^{(0)}\right\rangle$$

$$c_n(t) = \delta_{nk} - \frac{i}{\hbar}\int_0^t dt' \exp\left(i\left(E_n^{(0)} - E_k^{(0)}\right)t'/\hbar\right)\left\langle E_n^{(0)}\right\rvert\hat{H}_1(t')\left\lvert E_k^{(0)}\right\rangle + \ldots$$

Emission and Absorption of radiation

Quantized EM field

$$\hat{\mathbf{A}}(\mathbf{r}) = \sum_{\mathbf{k}\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0 V\omega}} \left(\hat{a}_{\mathbf{k}\lambda} \vec{\varepsilon}(\mathbf{k}\lambda) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \hat{a}^{+}_{\mathbf{k}\lambda} \vec{\varepsilon}^{\,*}(\mathbf{k}\lambda) e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \right)$$

$$\hat{H} = \sum_{\mathbf{k}\lambda} \hbar\omega \left(\hat{a}^{+}_{\mathbf{k}\lambda} \hat{a}_{\mathbf{k}\lambda} + \tfrac{1}{2} \right)$$

$$\left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}^{+}_{\mathbf{k}'\lambda'} \right] = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\lambda\lambda'}$$

Spontaneous photon emission rate from EM quantum fluctuations

$$\frac{1}{\tau} = \frac{\alpha\omega_0}{2\pi c^2} \int d\Omega_{\mathbf{k}} \sum_{\lambda} \left| \left\langle \psi_b \left| \frac{\hat{\mathbf{p}}}{m} e^{-i\mathbf{k}\cdot\mathbf{r}} \right| \psi_a \right\rangle \cdot \vec{\varepsilon}(\mathbf{k},\lambda) \right|^2$$

Dipole approximation:

$$\frac{1}{\tau} = \frac{\omega_0^3}{3\pi\varepsilon_0 \hbar c^3} \left| \left\langle \psi_b \left| e\hat{\mathbf{r}} \right| \psi_a \right\rangle \right|^2$$

Good Luck with the Exam!

Tuesday May 16, 9-noon in room 274