Maximal Supersymmetry
in
Light-Cone Superspace

Dmitry Belyaev
Institute for Fundamental Theory
University of Florida

November 17th, 2009
Seminar at JHU

Acknowledgements

This talk is based on my recent paper

- “Dynamical supersymmetry in maximally supersymmetric gauge theories” (arXiv: 0910.5471)

as well as on soon-to-appear-work in collaboration with

- Lars Brink (Chalmers University of Technology, Göteborg, Sweden)
- Sung-Soo Kim (Universite Libre de Bruxelles, Brussels, Belgium)
- Pierre Ramond (IFT, University of Florida, Gainesville, FL)

Outline

- Motivation: why "maximal supersymmetry" and
  why "light-cone superspace"?
- LC superfields and dynamical supersymmetry
- Fitting $d = 10$ $N = 1$ SYM into LC superspace (also $d = 4$ $N = 4$ SYM)
- Fitting $d = 3$ $N = 8$ BLG theory into LC superspace
- Bottom-up approach: ansatz, constraints and solutions
- Outlook: open problems

Why “maximal supersymmetry”?

- LHC: supersymmetry is still alive…
- Phenomenology: minimal supersymmetry (4 supercharges in $d = 4$)
- Fundamental theory: maximal supersymmetry (16 or 32 supercharges)
- String theory: (type I, heterotic) ⇒ 16, (type IIA, IIB) ⇒ 32
- M-theory: $d = 11$ supergravity (32 supercharges)
- Type IIB on $AdS_5 \times S^5$: $N = 4$ $d = 4$ SYM (16 supercharges)
- M-theory on $AdS_4 \times S^7$: $N = 8$ $d = 3$ SCS=BLG (16 supercharges)
- M-theory on $AdS_7 \times S^4$: $N = (2, 0)$ $d = 6$??? (16 supercharges)
- AdS/CFT: “AdS” ⇒ superconformal ⇒ $16 + 16$ supercharges

Why “light-cone superspace”?

- Superspace: makes supersymmetry manifest
- The usual (off-shell) superspace: [Salam, Strathdee 1974]
  - need auxiliary fields
  - great for $N = 1$, complicated for $N > 1$
  - nonexistent(?) for maximal N [Siegel, Roček 1981]
- Light-cone (on-shell) superspace: [Siegel, Gates 1981]
  - uses only physical degrees of freedom
  - makes only half supersymmetry manifest
  - Lorentz invariance is not manifest
  - great (manageable) for maximal N [Brink, Lindgren, Nilsson 1983]
  - useful for quantum calculations [Mandelstam 1983]
    (was used to prove UV finiteness of $d = 4$ $N = 4$ SYM)

Light cone

Light-cone superspace basics

Light-cone coordinates

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^3), \quad \partial_{\pm} = \frac{\partial}{\partial x^{\pm}}, \quad \partial^{\pm} = -\partial_{\mp}$$

Choose x^+ as light-cone "time" coordinate ⇒ ∂^- is "time derivative."

Half of 16 supersymmetries manifest ⇒ 8 Grassmann coordinates:

$$\theta^m, \quad \bar{\theta}_m = (\theta^m)^*, \quad m = 1, 2, 3, 4$$

This makes $SU(4)$ R-symmetry manifest. { Supergravity ⇒ $SU(8)$ }

Kinematical supersymmetry generators (q's) are

$$q^m = -\frac{\partial}{\partial \bar{\theta}_m} + \frac{i}{\sqrt{2}}\theta^m \partial^+, \quad \bar{q}_m = \frac{\partial}{\partial \theta^m} - \frac{i}{\sqrt{2}}\bar{\theta}_m \partial^+$$

Superspace covariant derivatives (d's) are

$$d^m = -\frac{\partial}{\partial \bar{\theta}_m} - \frac{i}{\sqrt{2}}\theta^m \partial^+, \quad \bar{d}_m = \frac{\partial}{\partial \theta^m} + \frac{i}{\sqrt{2}}\bar{\theta}_m \partial^+$$

Light-cone superfields

For maximally supersymmetric gauge theories, physical (on-shell) degrees of freedom are described by a LC superfield ϕ_a satisfying the chirality constraint

$$d^m \phi_a = 0$$

and an extra reality (“inside-out”) constraint

$$\overline{\phi}_a \equiv \phi_a^* = \frac{\bar{d}_1 \bar{d}_2 \bar{d}_3 \bar{d}_4}{2\partial^{+2}} \phi_a \quad \Leftrightarrow \quad \bar{d}_{mn}\phi_a = \frac{1}{2}\varepsilon_{mnpq} d^{pq} \overline{\phi}_a$$

In the chiral “y-basis” with $y^- = x^- - \frac{i}{\sqrt{2}}\theta^m \bar{\theta}_m$ and $\partial^+ = -\frac{\partial}{\partial y^-}$

$$\phi_a = \frac{1}{\partial^+} A_a + \theta^m \frac{1}{\partial^+} \bar{\chi}_{ma} + \theta^{mn} \bar{C}_{mna} + \theta^{mnp} \varepsilon_{mnpq} \chi^q_a + \theta^{mnpq} \varepsilon_{mnpq} \partial^+ \overline{A}_a$$

Helicities are $(-1, -\frac{1}{2}, 0, +\frac{1}{2}, +1)$. “Inside-out” conditions are

$$\overline{A}_a = (A_a)^*, \quad \chi^m_a = (\bar{\chi}_{ma})^*, \quad (\bar{C}_{mna})^* = \frac{1}{2}\varepsilon^{mnkl} \bar{C}_{kla} \equiv C^{mn}_a$$

The d.o.f. count is: $1 + 6 + 1$ bosonic and $4 + 4$ fermionic.

Light-cone superfields

For maximally supersymmetric SUGRA (with 32 supercharges):

$$\theta^m, \quad \bar{\theta}_m = (\theta^m)^*, \quad m = 1, \ldots, 8 \quad \Rightarrow \quad SU(8)$$

$$d^m \phi = 0, \quad \bar{d}_{mnpq} \phi = \frac{1}{2} \varepsilon_{mnpqrstu} d^{rstu} \overline{\phi}$$

$$\phi = \frac{1}{\partial^{+2}} h + \theta^m \frac{1}{\partial^{+2}} \bar{\psi}_m + \theta^{mn} \frac{1}{\partial^+} \bar{A}_{mn} + \theta^{mnp} \frac{1}{\partial^+} \bar{\chi}_{mnp} + \theta^{mnpq} \bar{C}_{mnpq}$$
$$+ \varepsilon_{mnpqrstu} \left[\theta^{mnpqr} \chi^{stu} + \theta^{mnpqrs} \partial^+ A^{tu} + \theta^{mnpqrst} \partial^+ \psi^u + \theta^{mnpqrstu} \partial^{+2} \overline{h} \right]$$

Helicities are $(-2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, +\frac{1}{2}, +1, +\frac{3}{2}, +2)$.

$$\overline{h} = h^*, \quad \psi^m = (\bar{\psi}_m)^*, \quad A^{mn} = (\bar{A}_{mn})^*, \quad \chi^{mnp} = (\bar{\chi}_{mnp})^*$$
$$(\bar{C}_{mnpq})^* = \frac{1}{2} \varepsilon^{mnpqrstu} \bar{C}_{rstu} \equiv C^{mnpq}$$

The d.o.f. count is: $(1 + 28 + 70 + 28 + 1) + (8 + 56 + 56 + 8) = 128 + 128$

Kin. and Dyn. supersymmmetry

Kinematical supersymmetry acts linearly:

$$\delta_{\text{k.s.}}\phi_a = (\zeta^m \bar{q}_m - \bar{\zeta}_m q^m)\phi_a$$

$$\phi_a = \frac{1}{\partial^+}A_a + \frac{i}{\sqrt{2}}\theta^{mn}\bar{C}_{mna} + \frac{1}{12}\theta^{mnpq}\varepsilon_{mnpq}\partial^+\overline{A}_a$$
$$+i\theta^m\frac{1}{\partial^+}\bar{\chi}_{ma} + \frac{\sqrt{2}}{6}\theta^{mnp}\varepsilon_{mnpq}\chi^q_a$$

$$\begin{aligned}
\delta_{\text{k.s.}}A_a &= i\zeta^m\bar{\chi}_{ma} \\
\delta_{\text{k.s.}}\bar{\chi}_{ma} &= \sqrt{2}\bar{\zeta}_m\partial^+A_a + \sqrt{2}\zeta^n\partial^+\bar{C}_{mna} \\
\delta_{\text{k.s.}}\bar{C}_{mna} &= -i(\bar{\zeta}_m\bar{\chi}_{na} - \bar{\zeta}_n\bar{\chi}_{ma} + \varepsilon_{mnpq}\zeta^p\chi^q_a)
\end{aligned}$$

Dynamical supersymmetry is nonlinear:

$$\delta_{\text{d.s.}}\phi_a = \mathcal{O}_{\text{d.s.}}\phi_a + f_a{}^{bc}\cdot O(\phi^2) + f_a{}^{bcd}\cdot O(\phi^3) + \ldots$$

Super Yang-Mills

SYM: covariant formulation

Based on a Lie algebra:

$$\phi = \phi_a T^a, \quad [T^b, T^c] = f^{bc}{}_a T^a, \quad f^{bc}{}_a = f^{[bc]}{}_a, \quad f^{[bc}{}_g f^{d]g}{}_a = 0$$

Gauge multiplet in $d = 10$ is (A_{Ma}, λ_a) with gauge and susy transformations

$$\delta_\omega A_{Ma} = \partial_M \omega_a + f^{bc}{}_a A_{Mb} \omega_c \equiv D_M \omega_a, \quad \delta \lambda_a = f^{bc}{}_a \lambda_b \omega_c$$

$$\delta_\epsilon A_{Ma} = i\bar{\epsilon}\Gamma_M \lambda_a, \quad \delta_\epsilon \lambda_a = \frac{1}{2}\Gamma^{MN}\epsilon F_{MNa}$$

Majorana-Weyl conditions on 32-component spinors: ⇒ $8+8$ d.o.f.

$$\epsilon^T C = \epsilon^\dagger \Gamma_0, \quad \Gamma_{11}\epsilon = \epsilon; \quad \lambda_a^T C = \lambda_a^\dagger \Gamma_0, \quad \Gamma_{11}\lambda_a = \lambda_a$$

Closure of the supersymmetry algebra ⇒ equations of motion:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = v^M \partial_M + \delta_\omega, \quad v^M = -2i(\bar{\epsilon}_2 \Gamma^M \epsilon_1), \quad \omega_a = -v^M A_{Ma}$$

$$\Gamma^M D_M \lambda_a = 0, \quad D^M F_{MNa} + \frac{i}{2} f^{bc}{}_a \bar{\lambda}_b \Gamma_N \lambda_c = 0$$

Top-down reduction: algorithm

To identify the embedding of component fields into the LC superfield ϕ_a and to find the way ϕ_a transforms under the dynamical supersymmetry, we will

- impose the LC gauge
- use EOM to solve for dependent components
- find susy transformations of independent *bosonic* components (include compensating gauge transformations!!!)
- match kin. susy to identify A_a, C_{mna} and χ_{ma}
- use dyn. susy of A_a to guess dyn. susy of ϕ_a
- verify the guess

[DB 0910.5471]

SYM: the LC gauge

LC decomposition: $A_M = (A_-, A_+, A_I)$, $\Gamma_M = (\Gamma_-, \Gamma_+, \Gamma_I)$

$$\lambda = \lambda_+ + \lambda_-, \quad \lambda_\pm = P_\pm \lambda, \quad P_+ = -\frac{1}{2}\Gamma_+\Gamma_-, \quad P_- = -\frac{1}{2}\Gamma_-\Gamma_+$$

LC gauge: $A_{-a} = 0$. EOM then imply

$$\lambda_{a-} = \frac{1}{2\partial^+}(\Gamma_-\Gamma^I D_I \lambda_{a+}), \quad A_{+a} = -\frac{1}{\partial^{+2}}(D^I F_{I-a} + \frac{i}{2} f^{bc}{}_a \overline{\lambda}_b \Gamma_- \lambda_c)$$

Modified supersymmetry (to stay in the LC gauge):

$$\delta'_\epsilon A_{-a} = i\overline{\epsilon}\Gamma_-\lambda_a + \partial_-\omega_a = 0 \quad \Rightarrow \quad \omega_a = \frac{i}{\partial^+}(\overline{\epsilon}_+\Gamma_-\lambda_{a+})$$

The independent bosonic components transform as

$$\begin{aligned} \delta_{\epsilon_-} A_{Ia} &= i\overline{\epsilon}_-\Gamma_I\lambda_{a+} \\ \delta_{\epsilon_+} A_{Ia} &= \frac{i}{2\partial^+}(\overline{\epsilon}_+\Gamma_-\Gamma^J D_J \Gamma_I \lambda_{a+}) + f^{bc}{}_a \frac{1}{\partial^+}(\partial^+ A_{Ib} \cdot \omega_c) \end{aligned}$$

Conclusion: ϵ_- describes kin. susy, ϵ_+ describes dyn. susy

SYM: one-component spinors

Choose a representation for $\Gamma^M = (\Gamma^\mu, \Gamma^{\widehat{I}+3})$, $\mu = 0,1,2,3$, $\widehat{I} = 1,\ldots,6$

$$\Gamma^\mu = i\gamma^\mu \otimes \begin{pmatrix} \delta^m_n & 0 \\ 0 & \delta^n_m \end{pmatrix}, \quad \Gamma^{\widehat{I}+3} = i\gamma_5 \otimes \begin{pmatrix} 0 & \Sigma^{\widehat{I}mn} \\ \bar{\Sigma}_{\widehat{I}mn} & 0 \end{pmatrix}$$

$m = 1,2,3,4$ is $SU(4)$ index. $SO(6) \sim SU(4)$ and $(\mathbf{4}\times\mathbf{4})_{\mathrm{asym}} = \mathbf{6}$

$$\epsilon_- = \begin{pmatrix} \epsilon^m_- \\ \epsilon_{m-} \end{pmatrix}, \quad \epsilon_+ = \begin{pmatrix} \epsilon^m_+ \\ \epsilon_{m+} \end{pmatrix}$$

$$\epsilon^m_- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha^m \end{pmatrix}, \quad \epsilon_{m-} = \begin{pmatrix} -\bar{\alpha}_m \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^m_+ = \begin{pmatrix} 0 \\ 0 \\ \beta^m \\ 0 \end{pmatrix}, \quad \epsilon_{m+} = \begin{pmatrix} 0 \\ \bar{\beta}_m \\ 0 \\ 0 \end{pmatrix}$$

$(\alpha^m)^* = \bar{\alpha}_m$ $\qquad$ $(\beta^m)^* = \bar{\beta}_m$

SYM: the LC superfield

The eliminated fields of the gauge multiplet (A_{Ma}, λ_a) are

$$A_{\pm a} = \frac{1}{\sqrt{2}}(A_{0a} \pm A_{3a}), \quad \lambda_{-a} = P_- \lambda_a$$

Using the remaining components to define

$$A_a = \frac{1}{\sqrt{2}}(A_{1a} + iA_{2a}), \quad \bar{C}_{mna} = \frac{1}{\sqrt{2}} \bar{\Sigma}_{\widehat{I}mn} A_{(\widehat{I}+3)a}$$

$$\lambda_{a+} = \begin{pmatrix} \lambda^m_{a+} \\ \lambda_{ma+} \end{pmatrix}, \quad \lambda^m_{a+} = \begin{pmatrix} 0 \\ 0 \\ \chi^m_a \\ 0 \end{pmatrix}, \quad \lambda_{ma+} = \begin{pmatrix} 0 \\ \bar{\chi}_{ma} \\ 0 \\ 0 \end{pmatrix}$$

$$(\chi^m_a)^* = \bar{\chi}_{ma}$$

we find that kin. susy transformations are reproduced! (with $\zeta^m = \sqrt{2}\alpha^m$)

$$\phi_a = \frac{1}{\partial^+} A_a + \theta^m \frac{1}{\partial^+} \bar{\chi}_{ma} + \theta^{mn} \bar{C}_{mna} + \theta^{mnp} \varepsilon_{mnpq} \chi^q_a + \theta^{mnpq} \varepsilon_{mnpq} \partial^+ \overline{A}_a$$

SYM: dyn. susy on A_a

The LC derivatives are

$$\partial^\pm = -\partial_\mp, \quad \partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_3)$$

Combining transverse derivatives into

$$\partial = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2), \quad \bar{\partial}_{mn} = \frac{1}{\sqrt{2}}\bar{\Sigma}_{\hat{I}mn}\partial_{\hat{I}+3}$$

we find that dyn. susy variation of A_a is

$$\begin{aligned} \delta'_{\epsilon_+} A_a &= i\xi^m \frac{1}{\partial^+}\left(D\bar{\chi}_{ma} + f^{bc}{}_a \partial^+ A_b \cdot \frac{1}{\partial^+}\bar{\chi}_{mc}\right) \\ &+ i\bar{\xi}_m \frac{1}{\partial^+}\left(D^{mn}\bar{\chi}_{na} + f^{bc}{}_a \partial^+ A_b \cdot \frac{1}{\partial^+}\chi^m_c\right) \end{aligned}$$

where $\xi^m = -\sqrt{2}\beta^m$ and

$$D\bar{\chi}_{ma} = \partial\bar{\chi}_{ma} + f^{bc}{}_a A_b \bar{\chi}_{mc}, \quad D^{mn}\bar{\chi}_{ka} = \partial^{mn}\bar{\chi}_{ka} + f^{bc}{}_a C^{mn}_b \bar{\chi}_{kc}$$

SYM: dyn. susy on ϕ_a

We can rewrite the previous result as

$$\begin{aligned}
\delta_{\xi\overline{Q}} A_a &= i\xi^m \frac{1}{\partial^+}\Big(\partial\bar\chi_{ma} + f^{bc}{}_a \partial^+ (A_b \frac{1}{\partial^+}\bar\chi_{mc})\Big) \\
\delta_{\bar\xi Q} A_a &= i\bar\xi_m \frac{1}{\partial^+}\Big(\partial^{mn}\bar\chi_{na} + f^{bc}{}_a (C_b^{mn}\bar\chi_{nc} + \partial^+ A_b \cdot \frac{1}{\partial^+}\chi_c^m)\Big)
\end{aligned}$$

From $\delta_{\xi\overline{Q}} A_a$ and the linear part of $\delta_{\bar\xi Q} A_a$ we guess that

$$\delta_{\xi\overline{Q}}\phi_a = \xi^m \frac{1}{\partial^+}\Big(\partial\bar q_m \phi_a - \bar\partial_{mn} q^n \phi_a + f^{bc}{}_a (\partial^+\phi_b \cdot \bar q_m \phi_c)\Big)$$

Conjugating and using the “inside-out” constraint, we reproduce $\delta_{\bar\xi Q} A_a$ after projection to the lowest component. This is the result for $d = 10$ SYM.

The result for $d = 4$ $N = 4$ SYM follows by setting $\bar\partial_{mn} = 0$

$$\delta_{\xi\overline{Q}}\phi_a = \xi^m \frac{1}{\partial^+}\Big(\partial\bar q_m \phi_a + f^{bc}{}_a (\partial^+\phi_b \cdot \bar q_m \phi_c)\Big)$$

This reproduces earlier results by [Ananth, Brink, Ramond 0501079] and [Ananth, Brink, Kim, Ramond 0505234]

SYM: the Hamiltonian

Dynamical supersymmetries commute into the "Hamiltonian shift"

$$[\delta_{\epsilon\bar{Q}}, \delta_{\bar{\epsilon}Q}]\phi_a = i\sqrt{2}\bar{\epsilon}_k \epsilon^k \delta_{P^-}\phi_a$$

which is a sum of a "time-translation" and the equation of motion

$$\partial^- \phi_a = \delta_{P^-}\phi_a = \frac{\partial^I \partial_I}{\partial^+}\phi_a + O(f) + O(f^2), \quad \partial^I \partial_I = \partial\bar{\partial} + \bar{\partial}_{mn}\partial^{mn}$$

This EOM follows from a Hamiltonian

$$H = \int d^4x d^4\theta d^4\bar{\theta}\ h^{ab}\left[\bar{\phi}_a \frac{\partial^I \partial_I}{\partial^{+2}}\phi_b + O(f) + O(f^2)\right]$$

which turns out to be a quadratic form

$$H = \frac{\partial}{\partial\bar{\epsilon}_k}\frac{\partial}{\partial\epsilon^k}\int d^4x d^4\theta d^4\bar{\theta} h^{ab}(\delta_\epsilon\phi_a)^* \frac{1}{\partial^+}(\delta_\epsilon\phi_b)$$

thanks to the "inside-out" constraint ($\bar{\phi} = \bar{\Pi}\phi$) and requiring

$$f^{abc} \equiv f^{ab}{}_e h^{ec} \quad \Rightarrow \quad f^{abc} = f^{[abc]}$$

Bagger-Lambert-Gustavsson

BLG: covariant formulation

Based on a 3-Lie algebra:

$$\phi = \phi_a T^a, \quad [T^b, T^c, T^d] = f^{bcd}{}_a T^a, \quad f^{bcd}{}_a = f^{[bcd]}{}_a$$

Fundamental Identity:

$$f^{[efg}{}_d f^{c]db}{}_a = 0$$

The $d = 3$ BLG multiplet is $(X^I_a, \Psi_a, \widetilde{A}_\mu{}^b{}_a)$. D.o.f. count is:

- $X^I, \quad I = 3, \ldots, 10, \quad$ on-shell $\quad \Rightarrow \quad$ 8 bosonic
- $\Psi, \quad \Psi^T C = \Psi^\dagger \Gamma_0, \quad \Gamma_{012}\Psi = -\Psi, \quad$ on-shell $\quad \Rightarrow \quad$ 8 fermionic
- $\widetilde{A}_\mu, \quad \mu = 0, 1, 2, \quad$ Chern-Simons gauge field, on-shell $\quad \Rightarrow \quad$ 0 bosonic

Ψ is 32-component; (Γ^μ, Γ^I) are 32×32 gamma matrices for $d = 11$

$$\epsilon^T C = \epsilon^\dagger \Gamma_0, \quad \Gamma_{012}\epsilon = +\epsilon$$

$\Rightarrow$ 16 real independent parameters in 32-component ϵ

BLG: covariant formulation

Gauge transformations

$$\delta_\Lambda X^I_a = X^I_b \widetilde{\Lambda}^b{}_a, \quad \delta_\Lambda \Psi_a = \Psi_b \widetilde{\Lambda}^b{}_a, \quad \delta_\Lambda \widetilde{A}_\mu{}^b{}_a = D_\mu \widetilde{\Lambda}^b{}_a$$

Supersymmetry transformations

$$\begin{aligned} \delta_\epsilon X^I_a &= i\bar{\epsilon}\Gamma^I \Psi_a, \quad \delta_\epsilon \widetilde{A}_\mu{}^b{}_a = i\bar{\epsilon}\Gamma_\mu \Gamma^I X^I_c \Psi_d f^{cdb}{}_a \\ \delta_\epsilon \Psi_a &= \Gamma^\mu \Gamma^I \epsilon D_\mu X^I_a - \frac{1}{6} X^I_b X^J_c X^K_d f^{bcd}{}_a \Gamma^{IJK} \epsilon \end{aligned}$$

The closure of the supersymmetry algebra on $(X^I_a, \Psi_a, \widetilde{A}_\mu{}^b{}_a)$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = v^\mu \partial_\mu + \delta_\Lambda, \quad v^\mu = -2i(\bar{\epsilon}_2 \Gamma^\mu \epsilon_1), \quad \widetilde{\Lambda}^b{}_a = -i(\bar{\epsilon}_2 \Gamma^{IJ} \epsilon_1) X^I_c X^J_d f^{cdb}{}_a$$

requires the Fundamental Identity and the following equations of motion

$$\begin{aligned} &\Gamma^\mu D_\mu \Psi_a + \frac{1}{2} \Gamma^{IJ} \Psi_b X^I_c X^J_d f^{bcd}{}_a = 0 \\ &\widetilde{F}_{\mu\nu}{}^b{}_a + \varepsilon_{\mu\nu\lambda} \left(X^I_c D^\lambda X^J_d + \frac{i}{2} \overline{\Psi}_c \Gamma^\lambda \Psi_d \right) f^{cdb}{}_a = 0 \end{aligned}$$

Compare with SYM

Based on a Lie algebra:

$$\phi = \phi_a T^a, \quad [T^b, T^c] = f^{bc}{}_a T^a, \quad f^{bc}{}_a = f^{[bc]}{}_a, \quad f^{[bc}{}_g f^{d]g}{}_a = 0$$

Gauge multiplet in $d = 10$ is (A_{Ma}, λ_a) with gauge and susy transformations

$$\delta_\omega A_{Ma} = \partial_M \omega_a + f^{bc}{}_a A_{Mb} \omega_c \equiv D_M \omega_a, \quad \delta\lambda_a = f^{bc}{}_a \lambda_b \omega_c$$

$$\delta_\epsilon A_{Ma} = i\bar{\epsilon}\Gamma_M \lambda_a, \quad \delta_\epsilon \lambda_a = \frac{1}{2}\Gamma^{MN}\epsilon F_{MNa}$$

Majorana-Weyl conditions on 32-component spinors: ⇒ $8+8$ d.o.f.

$$\epsilon^T C = \epsilon^\dagger \Gamma_0, \quad \Gamma_{11}\epsilon = \epsilon; \quad \lambda_a^T C = \lambda_a^\dagger \Gamma_0, \quad \Gamma_{11}\lambda_a = \lambda_a$$

Closure of the supersymmetry algebra ⇒ equations of motion:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = v^M \partial_M + \delta_\omega, \quad v^M = -2i(\bar{\epsilon}_2 \Gamma^M \epsilon_1), \quad \omega_a = -v^M A_{Ma}$$

$$\Gamma^M D_M \lambda_a = 0, \quad D^M F_{MNa} + \frac{i}{2} f^{bc}{}_a \overline{\lambda}_b \Gamma_N \lambda_c = 0$$

BLG: the LC gauge

The $d = 3$ coordinates are (x^0, x^1, x^2) and we take

$$\partial_\pm = \frac{\partial}{\partial x^\pm}, \qquad x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1), \qquad \partial_2 = \frac{\partial}{\partial x^2}$$

LC gauge: $\widetilde{A}_-{}^b{}_a = 0$. EOM then imply

$$\begin{aligned} \widetilde{A}_+{}^b{}_a &= -\frac{1}{\partial^+}(X^I_c D_2 X^I_d + \frac{i}{2}\overline{\Psi}_c \Gamma_2 \Psi_d) f^{cdb}{}_a \\ \widetilde{A}_2{}^b{}_a &= \frac{1}{\partial^+}(X^I_c \partial^+ X^I_d - \frac{i}{2}\overline{\Psi}_{c+} \Gamma_- \Psi_{d+}) f^{cdb}{}_a \\ \Psi_{a-} &= \frac{1}{2\partial^+}\Gamma_-(\Gamma^2 D_2 \Psi_{a+} + \frac{1}{2}\Gamma^{IJ}\Psi_{b+} X^I_c X^J_d f^{bcd}{}_a) \end{aligned}$$

where

$$D_2 \Psi_{a+} = \partial_2 \Psi_{a+} - \Psi_{b+} \widetilde{A}_2{}^b{}_a$$

Only (X^I_a, Ψ_{a+}) remain as independent $\quad \Rightarrow \quad 8 + 8$ d.o.f.

BLG: modified supersymmetry

Compensating gauge transformations:

$$\delta'_\epsilon \widetilde{A}_-{}^b{}_a = 0 \quad \Rightarrow \quad \widetilde{\Lambda}^b{}_a = \frac{i}{\partial^+}(\bar{\epsilon}_+\Gamma_-\Gamma^I X^I_c \Psi_{d+})f^{cdb}{}_a$$

Modified supersymmetry for the independent bosonic components

$$\begin{aligned} \delta'_{\epsilon_-} X^I_a &= i\bar{\epsilon}_-\Gamma^I\Psi_{a+} \\ \delta'_{\epsilon_+} X^I_a &= i\bar{\epsilon}_+\Gamma^I\Psi_{a-} + X^I_b\widetilde{\Lambda}^b{}_a \end{aligned}$$

As in SYM, ϵ_- describes kin. susy and ϵ_+ describes dyn. susy

---

Recall that in SYM:

$$\begin{aligned} \delta_{\epsilon_-} A_{Ia} &= i\bar{\epsilon}_-\Gamma_I\lambda_{a+} \\ \delta_{\epsilon_+} A_{Ia} &= \frac{i}{2\partial^+}(\bar{\epsilon}_+\Gamma_-\Gamma^J D_J\Gamma_I\lambda_{a+}) + f^{bc}{}_a\frac{1}{\partial^+}(\partial^+ A_{Ib}\cdot\omega_c) \end{aligned}$$

BLG: the LC superfield

Spinorial projectors

$$P_+ = -\frac{1}{2}\Gamma_+\Gamma_- = \frac{1}{2}(1+\Gamma_0\Gamma_1), \quad P_- = -\frac{1}{2}\Gamma_-\Gamma_+ = \frac{1}{2}(1-\Gamma_0\Gamma_1)$$

Taking $\Gamma_2 = -\Gamma_{11}$ and using $\Gamma_{012}\epsilon = \epsilon$, $\Gamma_{012}\Psi = -\Psi$ gives

$$\Gamma_{11}\epsilon_- = +\epsilon_-, \quad \Gamma_{11}\Psi_{a+} = +\Psi_{a+}$$

$$\Gamma_{11}\epsilon_+ = -\epsilon_+, \quad \Gamma_{11}\Psi_{a-} = -\Psi_{a-}$$

The decomposition of ϵ_-, Ψ_{a+} is the same as of ϵ_-, λ_{a+} in SYM.

$$\Psi_{a+} \quad \Rightarrow \quad \bar{\chi}_{ma}; \quad A_a = \frac{1}{\sqrt{2}}(X_a^3 + iX_a^4), \quad \bar{C}_{mna} = \frac{1}{\sqrt{2}}\bar{\Sigma}_{\hat{I}mn}X_a^{\hat{I}+4}$$

$$\phi_a = \frac{1}{\partial^+}A_a + \theta^m\frac{1}{\partial^+}\bar{\chi}_{ma} + \theta^{mn}\bar{C}_{mna} + \theta^{mnp}\varepsilon_{mnpq}\chi_a^q + \theta^{mnpq}\varepsilon_{mnpq}\partial^+\overline{A}_a$$

Kinematical supersymmetry transformations match.

BLG: dyn. susy on A_a

Using $\epsilon_+^T C = \epsilon_+^\dagger \Gamma_0$, $\Gamma_{11}\epsilon_+ = -\epsilon_+$, $P_+\epsilon_+ = \epsilon_+$ we find

$$\epsilon_+ = \begin{pmatrix}\epsilon_+^m \\ \epsilon_{m+}\end{pmatrix}, \quad \epsilon_+^m = \begin{pmatrix}0 \\ \eta^m \\ 0 \\ 0\end{pmatrix}, \quad \epsilon_{m+} = \begin{pmatrix}0 \\ 0 \\ \bar\eta_m \\ 0\end{pmatrix}, \quad \bar\eta_m = (\eta^m)^*$$

Combining the ingredients gives

$$\begin{aligned} \delta_{\bar\eta Q} A_a &= 2\bar\eta_m A_b \frac{1}{\partial^+}(A_c \chi_d^m - C_c^{mn}\bar\chi_{nd}) f^{bcd}{}_a \\ \delta_{\eta\overline{Q}} A_a &= -\eta^m \frac{\partial_2}{\partial^+}\bar\chi_{ma} + \eta^m \bar U_{mbcd} f^{bcd}{}_a \end{aligned}$$

$$\begin{aligned} \bar U_{mbcd} \equiv \frac{1}{\partial^+}\Big[& 2\bar\chi_{mb}\frac{1}{\partial^+}(\overline{A}_c\partial^+ A_d) - 2\partial^+ A_b \cdot \frac{1}{\partial^+}(\overline{A}_c\bar\chi_{md}) \\ & -2\bar\chi_{kb}\frac{1}{\partial^+}(\bar C_{mnc}\partial^+ C_d^{nk}) - 2\partial^+ A_b \cdot \frac{1}{\partial^+}(\chi_c^n \bar C_{mnd}) + i\sqrt{2}\bar\chi_{mb}\frac{1}{\partial^+}(\chi_c^n \bar\chi_{nd})\Big] \end{aligned}$$

BLG: dyn. susy on ϕ_a

Using $\delta_{\overline{\eta}Q} A_a$ and the linear part of $\delta_{\eta\overline{Q}} A_a$

$$\begin{aligned} \delta_{\overline{\eta}Q} A_a &= 2\bar{\eta}_m A_b \frac{1}{\partial^+}(A_c \chi^m_d - C^{mn}_c \bar{\chi}_{nd}) f^{bcd}{}_a \\ \delta_{\eta\overline{Q}} A_a &= -\eta^m \frac{\partial_2}{\partial^+} \bar{\chi}_{ma} + \eta^m \bar{U}_{mbcd} f^{bcd}{}_a \end{aligned}$$

we guess that

$$\delta_{\overline{\eta}Q} \phi_a = i\bar{\eta}_m q^m \frac{\partial_2}{\partial^+} \phi_a + 2i\bar{\eta}_m \frac{1}{\partial^+}(\partial^+ \phi_b \cdot \frac{1}{\partial^+} W^m_{cd}) f^{bcd}{}_a$$

$$W^m_{cd} = \partial^+ \phi_c \cdot \partial^+ d^m \overline{\phi}_d + \frac{i}{\sqrt{2}} d^{mn} \overline{\phi}_c \cdot \partial^+ \bar{d}_n \phi_d$$

Using the “inside-out” constraint, we have

$$\delta_{\eta\overline{Q}} \phi_a = \frac{\bar{d}_{[4]}}{2\partial^{+2}} \Big(\delta_{\overline{\eta}Q} \phi_a\Big)^*, \quad \bar{d}_{[4]} \equiv \frac{1}{4!} \varepsilon^{ijkl} \bar{d}_{ijkl}$$

Projecting to $\theta = 0$ we reproduce $\delta_{\eta\overline{Q}} A_a$ (after a few pages of algebra).

Bottom-up approach

$PSU(2,2|4)$ **on LC**

$d=4$ conformal group is $SO(4,2)\sim SU(2,2) \quad \Rightarrow \quad PSU(2,2|4) \quad \Rightarrow \quad SU(4)$

$$P^+=-i\partial^+,\quad P=-i\partial,\quad P^-=-i\frac{\partial\overline{\partial}}{\partial^+}$$

$$J^+=ix\partial^+,\quad J^{+-}=i(1+\xi),\quad J^-=-i\frac{\partial}{\partial^+}A$$

$$J=(x\overline{\partial}-\overline{x}\partial)-\tau,\quad D=i(\xi-(x\overline{\partial}+\overline{x}\partial))$$

$$K^+=2ix\overline{x}\partial^+,\quad K=2ixA,\quad K^-=2i\frac{1}{\partial^+}AB$$

$$J^m_n=T^m_n-\frac{1}{2}\tau\delta^m_n \qquad T^m_n=\frac{i}{2\sqrt{2}\partial^+}[q^m,\bar{q}_n],\quad \tau=\tfrac{1}{2}T^m_m$$

$$q^m=-\partial^m+\frac{i}{\sqrt{2}}\theta^m\partial^+,\quad Q^m=\frac{\overline{\partial}}{\partial^+}q^m$$

$$s^m=i\sqrt{2}\overline{x}q^m,\quad S^m=i\sqrt{2}\frac{1}{\partial^+}Aq^m$$

and $\bar{P}$, $\bar{J}^+$, $\bar{J}^-$, $\bar{K}$, $\bar{q}_m$, $\bar{Q}_m$, $\bar{s}_m$, $\bar{S}_m$ by conjugation.

$$A=\xi+\tau-x\overline{\partial},\quad B=\xi-\tau-\overline{x}\partial,\quad \xi=x^-\partial^+-\tfrac{1}{2}(\theta^m\bar{\partial}_m+\bar{\theta}_m\partial^m)$$

$OSp(2,2|8)$ on LC

$d = 3$ conformal group is $SO(3,2) \sim Sp(2,2) \quad \Rightarrow \quad OSp(2,2|8) \quad \Rightarrow \quad SO(8)$

$$P^+ = -i\partial^+, \quad P = -i\partial, \quad P^- = -i\frac{\partial}{2\partial^+}$$

$$J^+ = ix\partial^+, \quad J^{+-} = i(1+\xi), \quad J^- = -i\frac{\partial}{\partial^+}\mathcal{A}$$

$$J = -\tau, \quad D = i(\xi + \frac{1}{2} - x\partial) \qquad \mathcal{A} = \xi + \frac{1}{2} - \frac{1}{2}x\partial$$

$$K^+ = ix^2\partial^+, \quad K = 2ix\mathcal{A}, \quad K^- = 2i\frac{1}{\partial^+}\mathcal{A}(\mathcal{A} - \frac{1}{2})$$

$$J^m_n = T^m_n - \frac{1}{2}\tau\delta^m_n \qquad T^m_n = \frac{i}{2\sqrt{2}\partial^+}[q^m, \bar{q}_n], \quad \tau = \tfrac{1}{2}T^m_m$$

$$q^m = -\partial^m + \frac{i}{\sqrt{2}}\theta^m\partial^+, \quad Q^m = \frac{1}{\sqrt{2}}\frac{\partial}{\partial^+}q^m$$

$$s^m = ixq^m, \quad S^m = i\frac{1}{\partial^+}q^m\mathcal{A}$$

$$J^{mn} = \frac{i}{\sqrt{2}}q^m q^n \frac{1}{\partial^+}, \quad \bar{J}_{mn} = \frac{i}{\sqrt{2}}\bar{q}_m\bar{q}_n\frac{1}{\partial^+}$$

and $\bar{q}_m$, $\bar{Q}_m$, $\bar{s}_m$, $\bar{S}_m$ by conjugation. In $d = 3$, $x = \overline{x}$ and $\partial = \overline{\partial}$.

Basic commutators

Three generators (q^m, Q^m, K) determine the rest! For $OSp(2,2|4)$:

$$\{q^m, \bar{q}_n\} = -\sqrt{2}\delta^m_n P^+, \quad \{q^m, \bar{Q}_n\} = -\delta^m_n P, \quad \{Q^m, \bar{Q}_n\} = -\sqrt{2}\delta^m_n P^-$$

$$[K, q^m] = s^m, \quad [K, Q^m] = \sqrt{2} S^m$$

$$\{q^m, \bar{s}_n\} = -i\sqrt{2}\delta^m_n J^+, \quad \{s^m, \bar{s}_n\} = \sqrt{2}\delta^m_n K^+$$

$$[K^+, P^-] = 2i(J^{+-} - D)$$

$$\{Q^m, \bar{s}_n\} = -i\delta^m_n(J^{+-} - D) - \frac{1}{2}(J\delta^m_n + 2J^m_n), \quad \{Q^m, s^n\} = J^{mn}$$

$$[K, P] = 2iD, \quad [K, P^-] = -2iJ^-, \quad [K, J^-] = iK^-$$

Note that in the $PSU(2,2|4)$ case

$$[K, P] = 2(J + iD), \quad \{Q^m, s^n\} = 0$$

Find Q^m, conjugate, commute ⇒ P^- ⇒ Hamiltonian.

Commute with K to find S^m, J^- and K^-.

Interacting theory

Only dynamical generators are modified by interactions:

$$\begin{aligned}\delta_{\text{kin}}\phi_a &= \mathcal{O}_{\text{kin}}\phi_a \\ \delta_{\text{dyn}}\phi_a &= \mathcal{O}_{\text{dyn}}\phi_a + f^{bc}{}_a \cdot O(\phi^2) + f^{bcd}{}_a \cdot O(\phi^3) + \ldots\end{aligned}$$

For $d = 4$ $N = 4$ SYM:

$$\delta_{\xi\overline{Q}}\phi_a = \xi^m \frac{1}{\partial^+}\Big(\partial\bar{q}_m\phi_a + f^{bc}{}_a(\partial^+\phi_b \cdot \bar{q}_m\phi_c)\Big)$$

For $d = 3$ $N = 8$ BLG theory:

$$\delta_{\overline{\eta}Q}\phi_a = i\bar{\eta}_m q^m \frac{\partial_2}{\partial^+}\phi_a + 2i\bar{\eta}_m \frac{1}{\partial^+}(\partial^+\phi_b \cdot \frac{1}{\partial^+}W^m_{cd})f^{bcd}{}_a$$

$$W^m_{cd} = \partial^+\phi_c \cdot \partial^+ d^m\overline{\phi}_d + \frac{i}{\sqrt{2}}d^{mn}\overline{\phi}_c \cdot \partial^+\bar{d}_n\phi_d$$

Dyn. susy ansatz: $PSU(2,2|4)$

“Coherent states”:

$$\mathcal{E}_\epsilon = \exp\left(\frac{\epsilon^k \bar{q}_k}{\partial^+}\right), \quad E_\epsilon = \exp\left(\frac{\epsilon^k \bar{d}_k}{\partial^+}\right); \quad d^m E_\epsilon \phi = i\sqrt{2}\epsilon^m E_\epsilon \phi$$

$$\bar{q}_m = \bar{d}_m - i\sqrt{2}\bar{\theta}_m \partial^+ \quad \Rightarrow \quad \mathcal{E}_\epsilon \phi_b \cdot \mathcal{E}_\epsilon^{-1} \phi_c = E_\epsilon \phi_b \cdot E_\epsilon^{-1} \phi_c$$

Dyn. susy ansatz for $O(f^1)$ in $PSU(2,2|4)$:

$$\delta^{(1)}_{\epsilon Q} \phi_a = f^{bc}{}_a \frac{1}{\partial^{+A}} \left(E_\epsilon \partial^{+B} \phi_b \cdot E_\epsilon^{-1} \partial^{+C} \phi_c \right)_{|\text{linear in } \epsilon}$$

“Dimensional constraint” (from J^{+-} or D):

$$B + C = A + 1$$

The key constraint:

$$[\delta_{P^-}, \delta_{J^-}]\phi_a = 0 \quad \Rightarrow \quad A = B = C = 1 \quad \Rightarrow \quad f^{bc}{}_a = f^{[bc]}{}_a$$

At $O(f^2)$ requires the Jacobi identity.

Dyn. susy ansatz: $OSp(2,2|8)$

Dyn. susy ansatz for $O(f^1)$ in $OSp(2,2|8)$:

$$\delta^{(1)}_{\epsilon Q}\phi_a = f^{bcd}{}_a \sum_\alpha \rho_\alpha \frac{\varepsilon^{i_1\ldots i_{2-2\alpha} j_1\ldots j_{2+2\alpha}}}{(2-2\alpha)!(2+2\alpha)!} \frac{\partial}{\partial\eta^{i_1\ldots i_{2-2\alpha}}} \frac{\partial}{\partial\zeta^{j_1\ldots j_{2+2\alpha}}} \times$$

$$\times \frac{1}{\partial^{+A_\alpha}} \Big(E_\epsilon E_\eta \partial^{+B_\alpha}\phi_b \cdot E_\epsilon^{-1} E_\eta^{-1} \frac{1}{\partial^{+M_\alpha}} (E_\zeta \partial^{+C_\alpha}\phi_c \cdot E_\zeta^{-1}\partial^{+D_\alpha}\phi_d)\Big)_{|\text{linear in } \epsilon}$$

“Dimensional constraint” (from J^{+-} or D):

$$B_\alpha + C_\alpha + D_\alpha = A_\alpha + M_\alpha + 4$$

Coset $SO(8)/SU(4)$ generators J^{mn} require $\rho_\alpha \sim (-)^\alpha$ and recursion relation:

$$(A,B,M,C,D)_{\alpha+1} = (A,B,M,C,D)_\alpha + (-1,-1,+2,+1,+1)$$

with $\alpha = (-1/2,+1/2)$ or $\alpha = (-1,0,+1)$. The key constraint:

$$[\delta_{P^-},\delta_{J^-}]\phi_a = 0 \quad \Rightarrow \quad (A,B,M,C,D)_{-\frac{1}{2}} = (3,3,-2,1,1) \quad \oplus \quad f^{[bcd]}{}_a$$

(Agrees with the BLG solution.) At $O(f^2)$ requires the Fundamental Identity.

Summary and Outlook

- Maximal susy enjoys living in LC superspace
- The same constrained LC superfield does the job
- Dynamical supersymmetry encodes essential information
- (*all* information in the superconformal theories)
- Superfield Hamiltonian is a quadratic form. Implications?
- What about $d = 6$? ($M5$-branes) $\Rightarrow$ $OSp(4|6,2)$
- What about supergravity to all orders in κ?
- Quantum effects $\Rightarrow$ $\hbar$-dependent dynamical supersymmetry?

Thanks!