Generalized Ward Identities for Primordial Perturbations

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Inflationary Paradigm

Inflation = The stage of accelerated expansion with a graceful exit into FRW.

Inhomogeneities we observe today, including fluctuations in CMB, are seeded by quantum fluctuations during inflation.

It is important to understand what are the relatively model independent statements.
Single Field Inflation:

\[ S = \int d^4 x \sqrt{-g} \left( R - \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) \]

The inflation is driven by the time evolution of the scalar field.

The inflaton starts uphill and slowly rolls down the potential. Inflation ends when it reaches the bottom of the potential.
One of the most useful discriminating measure among various models of inflation is non-gaussianity.

In particular, it is important to understand the nature of the consistency conditions.
The Consistency Relation:

Maldacena '02, Creminelli and Zaldarriaga '04

\[ \lim_{q \to 0} \frac{\langle \zeta \bar{q} \zeta \bar{p} \zeta - \bar{q} - \bar{p} \rangle}{P_\zeta(q)} = - \left( 3 + p_k \frac{\partial}{\partial p_k} \right) P_\zeta(p) \]

Follows from symmetries

Holds in all models of single-field inflation where background is an attractor

Bunch-Davis initial state is assumed (?)

It is interesting because it could be violated
Residual Diffs as Infinite Number of Global Symmetries:

Hinterbichler, Hui, Khoury ’13

Infinite Symmetries $\rightarrow$ Infinite Number of Consistency Conditions

$$\lim_{q \to 0} \frac{\partial^n}{\partial q^n} \left( \frac{\langle \zeta(q)\zeta(p)\zeta(p') \rangle'}{P_\zeta(q)} + \frac{\langle \gamma(q)\zeta(p)\zeta(p') \rangle'}{P_\gamma(q)} \right) \sim \frac{\partial^n}{\partial p^n} \langle \zeta(p)\zeta(p') \rangle'$$

$n = 0, 1$: 3-point functions are completely fixed by symmetries  
Creminelli, Norena and Simonovic ’12

$n \geq 2$: only certain combinations of derivatives are constrained
Questions:

Why are the residual gauge symmetries useful?

What if we could fix the gauge completely?
The theory is highly constrained by the underlying gauge symmetry, even when it’s fixed.

Gauge symmetries lead to Slavnov-Taylor identities \textit{Slavnov ’72}

Certain consequences of the gauge symmetry can be re-derived using the residual symmetries; e.g. BRST.

The point is simple: the gauge-fixing term is not a problem, since it is pretty much model independent.
Warming-up with QED

Slavnov ’72

\[ Z[J_\mu, \eta, \bar{\eta}] = \int D A_\mu D \bar{\psi} D \psi e^{iS + iS_{g.-f.} + iS_{ext.}} \]

where

\[ S_{g.-f.} = -\frac{1}{2\xi} \int d^4x (\partial^\mu A_\mu)^2 \]

\[ S_{ext.} = \int d^4x \left( J^\mu A_\mu + \bar{\eta} \psi + \bar{\psi} \eta \right) \]

Under gauge transformation

\[ A_\mu \rightarrow A_\mu + \partial_\mu \Lambda; \quad \psi \rightarrow \psi - i\Lambda \psi \]
Generalized Ward-Takahashi Identity

\[
\left[ \frac{i
\Box}{\xi} \partial^\mu \frac{\delta}{\delta J^\mu} - \partial^\mu J^\mu - \bar{\eta} \frac{\delta}{\delta \bar{\eta}} + \eta \frac{\delta}{\delta \eta} \right] Z[J, \bar{\eta}, \eta] = 0
\]

This can be recast in terms of the effective action

\[-\frac{\Box}{\xi} \partial^\mu A_\mu + \partial^\mu \frac{\delta \Gamma}{\delta A_\mu} + i \psi \frac{\delta \Gamma}{\delta \psi} - i \bar{\psi} \frac{\delta \Gamma}{\delta \bar{\psi}} = 0\]
Generalized Ward-Takahashi Identity:

\[ q^\mu \Gamma^{A\bar{\psi}\psi}_\mu (q, p, -p - q) = \Gamma_\psi(p + q) - \Gamma_\psi(p) \]

In squeezed limit

\[ \Gamma^{A\bar{\psi}\psi}_\mu (0, p, -p) = \frac{\partial \Gamma_\psi(p)}{\partial p^\mu} \]

This relation works for fermionic as well as for scalar QED
The Most General Solution:

\[ \Gamma^{A\bar{\psi}\psi}_\mu(q, p, -p - q) = \sum_{n=0}^{\infty} q^n \frac{\partial^{n+1} \Gamma_\psi(p)}{\partial p_\mu \partial p^n} + C_\mu \]

with

\[ q^\mu C_\mu = 0 \]

Analyticity of vertex functional \( \Rightarrow \lim_{q \to 0} C_\mu = 0 \)

e.g. \( C^\mu = q_\nu [\gamma^\nu, \gamma^\mu] \), corresponding to the non-minimimal coupling

\[ F_{\mu\nu} \bar{\psi} \gamma^\mu \gamma^\nu \psi \]
Cosmology:

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)d\vec{x}^2; \quad \phi = \bar{\phi}(t)$$

Excitations of the background are introduced as

$$g_{\mu\nu} = \bar{g}_{\mu\nu}(t) + a^2(t)h_{\mu\nu}; \quad \phi = \bar{\phi}(t) + \varphi$$

Comoving gauge is given by $\varphi = 0$

Under spatial diffeomorphisms

$$h_{ij} \rightarrow h_{ij} + \partial_i \xi_j + \partial_j \xi_i + \xi^k \partial_k h_{ij} + h_{ik} \partial_j \xi^k + h_{jk} \partial_i \xi^k$$

The shift and lapse are considered to be integrated out
The Fixed-time Path-Integral Approach:
Goldberger, Hui and Nicolis ’13

\[ Z[J] = \int \mathcal{D}h_{ij} P[h, t] e^{\int d^3x \, h_{ij} J^{ij}} \]
Slavnov-Taylor Identities for Cosmological Perturbations:

LB, Justin Khoury ’13

\[ 2 \partial_j \left( \frac{1}{6} \delta_{ij} \frac{\delta \Gamma}{\delta \zeta} + \frac{\delta \Gamma}{\delta \gamma_{ij}} \right) = \partial_i \zeta \frac{\delta \Gamma}{\delta \zeta} + \ldots \]

\[ q_j \left( \frac{\delta_{ij}}{3} \Gamma^{\zeta\zeta\zeta} (\vec{q}, \vec{p}, -\vec{q} - \vec{p}) + 2 \Gamma^{\gamma\zeta\zeta}_{ij} (\vec{q}, \vec{p}, -\vec{q} - \vec{p}) \right) = \]

\[ q_i \Gamma_\zeta (\vec{p}) - p_i \left( \Gamma_\zeta (|\vec{q} + \vec{p}|) - \Gamma_\zeta (\vec{p}) \right) \]
General Solution:

\[
\frac{1}{3} \delta_{ij} \Gamma^{\zeta \zeta \zeta} (\vec{q}, \vec{p}, -\vec{q} - \vec{p}) + 2 \Gamma^{\gamma \zeta \zeta}_{ij} (\vec{q}, \vec{p}, -\vec{q} - \vec{p}) = K_{ij} + A_{ij}
\]

\(K_{ij}\) is a Taylor series in \(q\) and is determined by the power spectrum of short modes.

\(A_{ij}\) is an arbitrary transverse and symmetric matrix

\[
A_{ij} = \epsilon_{ikm} \epsilon_{j\ell n} q^k q^\ell \left( a(\vec{p}, \vec{q}) \delta^{mn} + b(\vec{p}, \vec{q}) p^m p^n \right)
\]

\(a(p, q)\) and \(b(p, q)\) are assumed to be regular in \(q \rightarrow 0\) limit
Analyticity Assumption:

Consistency relations hold if $A_{ij}$ starts to contribute at $q^2$ order

Analyticity/locality is a nontrivial assumption and holds only for adiabatic modes

$$N_i \supset -a^2 \frac{\dot{H}}{H^2} \frac{q_i}{q^2} \zeta$$
Correlation Functions:

\[
\langle \zeta \bar{q} \zeta \bar{p} \zeta - \bar{q} - \bar{p} \rangle' = P_\zeta(q)P_\zeta(p)P_\zeta(|\bar{q} + \bar{p}|)\Gamma^{\zeta\zeta\zeta}(\bar{q}, \bar{p}, -\bar{q} - \bar{p})
\]

\[
\langle \gamma \vec{q} \bar{q} \zeta \bar{p} \zeta - \bar{q} - \bar{p} \rangle' = \hat{P}^{ijk\ell}(\hat{q})P_\gamma(q)P_\zeta(p)P_\zeta(|\bar{q} + \bar{p}|)\Gamma^{\gamma\zeta\zeta}_{k\ell}(\bar{q}, \bar{p}, -\bar{q} - \bar{p})
\]
Consistency Relations:

3-point functions are determined by power spectrum up to order $q$

\[
\frac{\langle \zeta \bar{q} \zeta \bar{p} \zeta - \bar{q} - \bar{p} \rangle'}{P_\zeta(q)} = - \left( 3 + p_k \frac{\partial}{\partial p_k} \right) P_\zeta(p) \\
- \frac{1}{2} q_k \left( 6 \frac{\partial}{\partial p_k} - p_k \frac{\partial^2}{\partial p_a \partial p_a} + 2 p_a \frac{\partial^2}{\partial p_a \partial p_k} \right) P_\zeta(p) + O(q^2)
\]

\[
\frac{\langle \gamma \bar{q} \zeta \bar{p} \zeta - \bar{q} - \bar{p} \rangle'}{P_\gamma(q)} = - \frac{1}{2} \hat{P}^{ijkl}(\hat{q}) p_k \frac{\partial}{\partial p_\ell} P_\zeta(p) \\
+ \frac{1}{4} \hat{P}^{ijkl}(\hat{q}) q_m \left( p_m \frac{\partial^2}{\partial p_k \partial p_\ell} - 2 p_k \frac{\partial^2}{\partial p_\ell \partial p_m} \right) P_\zeta(p) + O(q^2)
\]
Higher Order Correlation Functions:

Higher order corrections are not uniquely fixed by symmetries. In particular, starting from $q^2$-order the consistency relation schematically looks like

$$\frac{\langle \gamma q \bar{\zeta} p \bar{\zeta} - \bar{q} - \bar{p} \rangle'}{P_\gamma(q)} + \frac{\langle \bar{\zeta} \bar{q} \bar{\zeta} \bar{p} \bar{\zeta} - \bar{q} - \bar{p} \rangle'}{P_\zeta(q)} = \text{Terms Determined by 2-point func.} + A - \text{term}$$

Model dependent pieces need to be projected out
Recovering Consistency Conditions to all Orders:

\[
\lim_{q \to 0} \text{Proj} \times \frac{\partial^n}{\partial q^n} \left( \frac{\langle \zeta(q)\zeta(p)\zeta(p') \rangle'}{P_\zeta(q)} + \frac{\langle \gamma(q)\zeta(p)\zeta(p') \rangle'}{P_\gamma(q)} \right) \\
\sim \text{Proj} \times \frac{\partial^n}{\partial p^n} \langle \zeta(p)\zeta(p') \rangle'
\]

These relations have been checked up to \( n = 3 \) explicitly, including those with hard tensor modes, for models with arbitrary \( c_s \). LB, Justin Khoury, Junpu Wang '14
Revisiting Analyticity

$\Gamma$ in 3D path-integral formalism is not quite the same as the one in 4D formalism

$$\Gamma^{3d}(\vec{q}, \vec{p}, -\vec{p} - \vec{q}) \sim \int d\tau d\tau_1 d\tau_2 \frac{P_\zeta(q, t, \tau)}{P_\zeta(q, t)} \frac{P_\zeta(p, t, \tau_1)}{P_\zeta(p, t)} \times$$

$$\frac{P_\zeta(|\vec{p} + \vec{q}|, t, \tau_2)}{P_\zeta(|\vec{p} + \vec{q}|, t)} \Gamma^{4d}(\vec{q}, \tau; \vec{p}, \tau_1; -\vec{p} - \vec{q}, \tau_2)$$

All of the above-mentioned results can be re-derived in the framework of 4D in-in path-integral
Modified Initial State

For arbitrary initial state invariant under spatial diffeomorphisms

\[
q_j \left( \frac{\delta_{ij}}{3} \Gamma^{\zeta \zeta \zeta} (\vec{q}, \vec{p}, -\vec{q} - \vec{p}) + 2 \Gamma_{ij}^{\gamma \zeta}(\vec{q}, \vec{p}, -\vec{q} - \vec{p}) \right) =
\]

\[
q_i \Gamma(\vec{p}) - p_i \left( \Gamma(\vec{q} + \vec{p}) - \Gamma(\vec{p}) \right)
\]

\[
\frac{1}{3} \delta_{ij} \Gamma_{\zeta \zeta \zeta} (\vec{q}, \vec{p}, -\vec{q} - \vec{p}) + 2 \Gamma_{ij}^{\gamma \zeta}(\vec{q}, \vec{p}, -\vec{q} - \vec{p}) = K_{ij} + A_{ij}
\]
Modified Initial State

The analyticity of $A_{ij}$ in $q \rightarrow 0$ limit cannot be argued any more.

$$\Gamma^{3d}(\vec{q}, \vec{p}, -\vec{p} - \vec{q}) \sim \int d\tau d\tau_1 d\tau_2 \frac{P_\zeta(q, t, \tau)}{P_\zeta(q, t)} \frac{P_\zeta(p, t, \tau_1)}{P_\zeta(p, t)} \times$$

$$\frac{P_\zeta(|\vec{p} + \vec{q}|, t, \tau_2)}{P_\zeta(|\vec{p} + \vec{q}|, t)} \Gamma^{4d}(\vec{q}, \tau; \vec{p}, \tau_1; -\vec{p} - \vec{q}, \tau_2)$$

The information about initial state contributes in $\Gamma^{4d}$ in the form of the terms localized at the initial time.
The scalar and tensor modes must start out from the same state.

\[ \psi = N \exp \left( \frac{1}{2} \int d^3 k \ f(k) \left[ 8 \zeta_k \zeta_{-k} - \gamma^{ij}_k \gamma^{ij}_{-k} \right] \right) \]
Summary:

We have shown that our approach recovers the known consistency conditions for primordial perturbations; as it should have been expected.

Our derivation makes precise the assumptions underlying the consistency relations, namely the regularity of the effective action in the $q \to 0$ limit; possible ways for violating these relations include the modified initial state LB, Justin Khoury, to appear.

Whether our approach can be used to derive the stronger consistency conditions than the ones presented here remains to be seen.