

Spinor Helicity Method

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Abstract

This is a writeup of the lecture given at the High Energy Theory Graduate Student seminar in the Fall, 2001 semester at Johns Hopkins University. I will discuss the motivation for using the Spinor Helicity Method when doing calculations in particle physics, such as the calculation in my thesis at Rochester, but try to talk more about the ideas behind the method and give a few general examples.

1 Introduction and Motivation

The Spinor Helicity Method is a very powerful technique developed independently by two groups that I know of [1, 2]. It allows for an efficient and elegant way to calculate matrix elements (and therefore cross sections) in complicated particle physics interactions involving heavy particles. Its power was originally intended for performing ugly calculations with W^\pm and Z^0 bosons, but more recently it has found a home in top phenomenology, where the ridiculously large top mass provides an ideal application. This method was the key point in my calculation that I did for a senior thesis at Rochester.

Problems in particle physics are described and ultimately solved using the techniques from Quantum Field Theory (QFT). Consider a generic interaction such as

$$ab \rightarrow cd$$

that is described (to lowest order) by the Feynman Diagram in Figure 1. Diagrams of this type, using the Feynman rules of Quantum Electrodynamics (QED), allow for a quick and easy calculation of quantum amplitudes for processes (traditionally denoted by $\mathcal{M}(ab \rightarrow cd)$ for “matrix element”) which correspond to the nonrelativistic scattering amplitudes in elementary quantum mechanics. As in that case, you can describe the “scattering cross section” using Fermi’s Golden Rule:

$$d\sigma = \mathcal{F} \cdot |\mathcal{M}|^2 \cdot d\Pi_n \tag{1}$$

where \mathcal{F} is some initial flux and $d\Pi_n$ is the n-dimensional Lorentz invariant phase space, given by:

$$d\Pi_n \equiv \prod_{i=1}^n \frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_i} \tag{2}$$

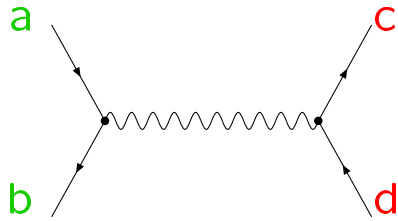


Figure 1: Tree-level Feynman diagram for the interaction $ab \rightarrow cd$.

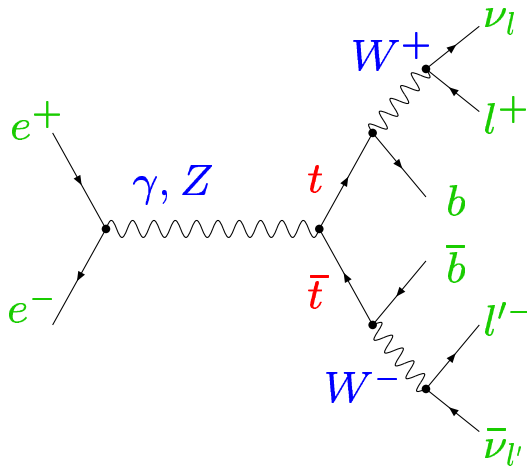


Figure 2: Tree-level Feynman diagram for top production and decay (dilepton event).

Further, in particle physics experiments, we do not have control over the spins of our particles (unless we specifically start with a polarized beam, which is usually not the case). In such cases, we must average over the initial spins and sum over the final spins:

$$|\mathcal{M}|^2 \Rightarrow \langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{spins} |\mathcal{M}|^2$$

where the $\frac{1}{4}$ comes in for the case of two initial particles with spin- $\frac{1}{2}$. It is here where many of the troubles begin. When you sum and square the matrix element, you can use the completeness of the Dirac spinors and the sum to make a trace over γ matrices dotted into the relevant 4-momenta. For simple scattering problems such as the “ $ab \rightarrow cd$ ” process considered above, this is a standard QFT homework problem. However, in the case of higher order diagrams, or diagrams with several vertices even at tree level, this can be a real nightmare to calculate by hand, and even the computer can have trouble grinding through the steps.

As a more concrete example, consider the case of top production and decay at the future Next Linear Collider (NLC); see Figure 2. A few important points should be made about the top quark [3]. Firstly, the top mass is impressively huge, currently measured to be $m_t c^2 = 175\text{GeV}$, with a corresponding width of $\Gamma_t = 1.55\text{GeV}$ and lifetime $\tau_t \sim 10^{-24}\text{s}$. With such an incredibly short lifetime, the top quark is unique in that it does not have time to hadronize and form QCD bound

states, even though it does have time to undergo gluon bremsstrahlung, which is important when considering higher-order diagrams.

Since the top lifetime is so short, the top quark need never be on-shell. Invoking the on-shell approximation is a good approximation to $(\frac{\Gamma_t}{m_t})^2$, or one part in 10^4 . This suggests that at tree level, we can afford to make this assumption, but at higher orders, this approximation breaks down. This presents a problem, since it means that one must calculate the amplitude that corresponds to the entire diagram in Figure 2, and not reduce the diagram into production and decay pieces:

$$\mathcal{M}(e^+e^- \rightarrow t\bar{t} \rightarrow bW^+\bar{b}W^-) \neq \mathcal{M}(e^+e^- \rightarrow t\bar{t}) \times \mathcal{M}(t \rightarrow bW^+) \times \mathcal{M}(\bar{t} \rightarrow \bar{b}W^-)$$

Therefore, the tree level Feynman diagram is an irreducible 6 vertex diagram with two top quarks. \mathcal{M} , and thence $\langle |\mathcal{M}|^2 \rangle$ is hell-spawned! See [4] for an analytic expression of \mathcal{M} .

But perhaps there is an easier way: what if we were to calculate \mathcal{M} directly as a complex number instead of $\langle |\mathcal{M}|^2 \rangle$? Then we would be dealing with spinor products to calculate as opposed to vector products from the traces, which are notoriously more difficult, even when handled numerically by a computer. Is there a way to do this that does not lose any information and makes our lives easier? The answer is yes.

2 Preliminaries and the Massless Case¹

In this talk, I will consider only fermions. As was mentioned above, the method can also be used to describe Gauge bosons, and was in fact originally intended to do so, but I will not discuss that here. For more details on how to apply these methods to the W^\pm or Z^0 , see [1].

The analysis begins by considering the standard completeness relation for a general Dirac spinor:

$$\sum_{\lambda=\pm} u_\lambda(p)\bar{u}_\lambda(p) = \not{p} + m \tag{3}$$

$$u_\lambda(p)\bar{u}_\lambda(p) = \frac{1}{2}(1 + \lambda\gamma^5) \not{p} \equiv \omega_\lambda \not{p} \tag{4}$$

where $u(p)$ can denote either a particle or antiparticle, and ω_λ is the spin projection operator in the massless case ($\lambda = \pm 1$). To proceed with this analysis, we promote Equations 3-4 to a fundamental level; i.e.: any objects $u_\lambda(p)$ that obey Equation 3 and 4 are legal spinors. We further assume for now that $p^2 = 0$, so that $m = 0$ in Equation 3.

With that in mind, let us derive expressions for spinor products in the massless limit, where we are allowed to use the Helicity spin-basis. Start by defining two new 4 vectors k_0 and k_1 that are more or less arbitrary but have the following properties:

$$k_0^2 = 0 \quad k_1^2 = -1 \quad k_0 \cdot k_1 = 0 \tag{5}$$

The arbitrariness of your choices corresponds to a change in phase that does not affect your final answer. Once you have defined your k-vectors, define the following spinors:

¹Throughout this talk, I will be following the notation of Kleiss and Stirling [1].

$$u_-(k_0)\bar{u}_-(k_0) \equiv \omega_- \not{k}_0 \quad (\text{negative helicity}) \quad (6)$$

$$u_+(k_0) \equiv \not{k}_1 u_-(k_0) \quad (\text{positive helicity}) \quad (7)$$

These are the “fundamental spinors”, since all the real spinors will be constructed from them. One can easily prove that these spinors satisfy Equation 4 for $m \equiv 0$:

$$u_+(k_0)\bar{u}_+(k_0) = \not{k}_1 u_-(k_0)\bar{u}_-(k_0) \not{k}_1 = \not{k}_1 \omega_- \not{k}_0 \not{k}_1 = -\omega_+ \not{k}_0 k_1^2 = \omega_+ \not{k}_0 \quad \mathbf{QED.}$$

That they satisfy Equation 3 is trivial.

Using this machinery, we can generally define the spinor for a massless particle:

$$\boxed{u_\lambda(p) = \not{p} u_{-\lambda}(k_0) \frac{1}{\sqrt{2p \cdot k_0}} \quad (p^2 = 0)} \quad (8)$$

Proof:

$$u_\lambda(p)\bar{u}_\lambda(p) = \not{p} u_{-\lambda}(k_0)\bar{u}_{-\lambda}(k_0) \not{p} \frac{1}{2p \cdot k_0} = \not{p} \omega_{-\lambda} \not{k}_0 \not{p} \frac{1}{2p \cdot k_0} = \omega_\lambda \not{p} \quad \mathbf{QED.}$$

Before proceeding further, it is useful to make a special note. When defining k_0 and k_1 , I mentioned that they are arbitrary up to Equations 5. However, from Equation 8, it can be seen that you should be careful that $p \cdot k_0 \neq 0$. In practice, you should be careful not to let $p \cdot k_0$ even approach zero. This is usually quite easy to do.

We now have all the tools we need to calculate matrix elements using this method. We give names to all the possible spinor products that might appear in our matrix element:

$$s(p_1, p_2) \equiv \bar{u}_+(p_1)u_-(p_2) = -s(p_2, p_1) \quad (9)$$

$$t(p_1, p_2) \equiv \bar{u}_-(p_1)u_+(p_2) = s^*(p_2, p_1) \quad (10)$$

$$d(p_1, p_2) \equiv 2p_1 \cdot p_2 = |s(p_1, p_2)|^2 \quad (11)$$

Notice that there is actually only one relevant product, which we call $s(p_1, p_2)$. It can be generally calculated:

$$\begin{aligned} s(p_1, p_2) &= \bar{u}_-(k_0) \not{p}_1 \not{p}_2 u_+(k_0) \frac{1}{\sqrt{4(p_1 \cdot k_0)(p_2 \cdot k_0)}} \\ &= \text{Tr}[\omega_- \not{k}_0 \not{p}_1 \not{p}_2 \not{k}_1] \frac{1}{\sqrt{4(p_1 \cdot k_0)(p_2 \cdot k_0)}} \\ &= [(p_1 \cdot k_0)(p_2 \cdot k_1) - (p_1 \cdot k_1)(p_2 \cdot k_0) - i\epsilon_{\mu\nu\rho\sigma} k_0^\mu k_1^\nu p_1^\rho p_2^\sigma] \frac{1}{\sqrt{4(p_1 \cdot k_0)(p_2 \cdot k_0)}} \end{aligned}$$

Now we will chose for concreteness the vectors $k_0 = (1; 1, 0, 0)$ and $k_1 = (0; 0, 1, 0)$. Note that these trivially satisfy Equations 5, and if we let the momenta of the initial particle be in the z-direction, and final particles have momenta in random directions, $p \cdot k_0 \neq 0$. Then we have:

$$s(p_1, p_2) = (p_1^y + ip_1^z) \sqrt{\frac{p_2^0 - p_2^x}{p_1^0 - p_1^x}} - (p_2^y + ip_2^z) \sqrt{\frac{p_1^0 - p_1^x}{p_2^0 - p_2^x}} \quad (12)$$

The proof is left to the reader. Note that immediately we have $t(p_1, p_2) = s^*(p_1, p_2)$, and $d(p_1, p_2) = |s(p_1, p_2)|^2 = 2p_1 \cdot p_2$. This method has given us the vector product along the way.

Two comments are in order before we move onto the massive case. First of all, note that in general, for n momenta we have defined an $n \times n$ matrix: $s(p_i, p_j)$. Once this matrix is known, all other calculations will involve simple multiplication and addition of s , t and d elements (all related to s). The role of the momentum becomes simple matrix indices. Secondly, notice that s is numerically stable for p_i and p_j colinear. This means that QCD and electroweak divergences from massless momenta cancel early in the calculation! See [1] for more details on this bonus.

3 Massive Fermions

Up to this point, all the spinors we have been considering have been massless. This is a good assumption for most fermions when compared to the W^\pm or Z^0 bosons, but this is not a valid assumption if you intend to study the top quark. Somehow we must generalize the above analysis to construct a massive spinor. The key is (once again) to treat the spin sum in Equation 3 and 4 as a fundamental result, and construct an object from the massless spinors that preserves the spin sum (with the mass nonzero). We will try:

$$u(q, s) = (\not{q} + m)u_-(k) \frac{1}{\sqrt{2q \cdot k}} \quad (13)$$

where $k^2 = 0$, and s is the spin vector. Then I claim that this is a good spinor that describes a particle of momentum q , mass m and spin s .

Proof:

$$\begin{aligned} u(q, s)\bar{u}(q, s) &= (\not{q} + m)\omega_- \not{k}(\not{q} + m) \frac{1}{2q \cdot k} \\ &= [(\not{q} + m) \not{k} + \gamma^5(\not{q} - m) \not{k}](\not{q} + m) \frac{1}{4q \cdot k} \\ &= [2q \cdot k + 2q \cdot k \gamma^5 - 2m\gamma^5 \not{k}](\not{q} + m) \frac{1}{4q \cdot k} \\ &= \frac{1}{2}(1 + \not{s}\gamma^5)(\not{q} + m) \end{aligned}$$

where

$$s \equiv \frac{1}{m}q - \frac{m}{q \cdot k}k \quad (14)$$

It is clear from Equation 14 that s is a legal spin 4-vector, i.e.: $s^2 = -1$ and $s \cdot q = 0$. Therefore, Equation 13 is a good spinor that describes a particle of momentum q , mass m and spin s . **QED.**

Now that we have defined our general massive spinor, decompose q into two massless momenta:

$$q \equiv p_1 + p_2 \quad p_i^2 = 0 \quad (15)$$

Now let k above be the massless p_2 , and note that $|s(p_1, p_2)| = \sqrt{2p_1 \cdot p_2} = m$. Put all of these together, we have the following general results (antiparticles considered too):

$$u(q, +s) = \alpha u_+(p_1) + u_-(p_2) \quad (16)$$

$$u(q, -s) = \beta u_-(p_1) + u_+(p_2) \quad (17)$$

$$v(q, +s) = \alpha u_+(p_1) - u_-(p_2) \quad (18)$$

$$v(q, -s) = \beta u_-(p_1) - u_+(p_2) \quad (19)$$

where

$$\alpha = \frac{s(p_1, p_2)}{m} \quad \beta = \frac{t(p_1, p_2)}{m} \quad (20)$$

are complex phases. Note also that in this basis, our spin vector is:

$$s = \frac{p_1 - p_2}{m}$$

Also notice that our values of α and β depend on our choice for k_0 and k_1 . This is where the arbitrariness of these vectors makes its appearance.

In summary, we can always express massive spinors in terms of two massless spinors of opposite helicity, and carry out our method for calculating \mathcal{M} as before.

4 Conclusions

When working out a process in particle physics, one must calculate $\langle |\mathcal{M}|^2 \rangle$, the matrix element squared and summed over spins. But this can lead to very complicated and difficult expressions that involve traces and vector products. We would like to avoid this when possible, especially when looking at higher order processes where \mathcal{M} is notoriously complicated. A better way to perform the calculation would be to consider \mathcal{M} directly as a complex number and then square it numerically and sum over spins afterwards. It is considerably easier to write an algorithm to accomplish this than to try and sort out the traces and keep track of vector products. Furthermore, if you make a wise choice for your basis $\{k_0, k_1\}$, your life can be made quite simpler! When performing this method, tricky divergence-cancellations occur before you start squaring, and hence are easier to see and understand, while avoiding much number crunching!

The key in all of these methods is the involvement of a massive particle that renders the others effectively massless. Originally these were the W^\pm and Z^0 bosons, but for my research, it was the top quark. Applying this method to other vector bosons such as the γ , W^\pm and Z^0 is discussed in [1].

Finally, a word or two should be mentioned about the phases that appear throughout this analysis. The real trick in this method is to chose your basis vectors shrewdly as to make your life much easier. In practice, this was the real challenge in my senior thesis. There were several times

that my choice of basis was different than the choice that was coded in the original FORTRAN code, for reasons of making my calculations easier. This was ok, but it meant that I had to compensate by performing an $SU(2)$ rotation on my result back to the program's default basis. It was this step that took a semester for me to learn how to do! See [4] for more details on how this was accomplished.

References

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- [3] For discovery papers, see F. Abe, et al. (CDF Collaboration), *Phys Rev Lett* **74**, 2626 (1995); S. Abachi, et al. (DØ Collaboration), *Phys Rev Lett* **74**, 2632 (1995).
- [4] A. E. Blechman, *Spin Correlations in Top Production and Decay at e^+e^- Colliders* (Senior Thesis), University of Rochester (2001).