171.105 Final Exam
Dec. 17, 2002 - Solutions

1. Change in potential energy goes into kinetic

$$\frac{1}{2} \cancel{m} v^2 = \cancel{m} g L \qquad v = \sqrt{2gL}$$

$$\omega = \frac{v}{L} = \sqrt{\frac{2g}{L}}$$

2. The bougant force is $0.9 \rho_0 V g$ where V is the cube's volume. This must equal the gravitational force $\rho V g$

$$\rho V g = 0.9 \rho_0 V g \quad \Rightarrow \quad \rho = 0.9 \rho_0$$

3. The response will be very small at low and high ω and peak near $\omega_0 = \sqrt{k/m}$. The width of the peak increases with the amount of damping

4. Initial $\quad K_i = \frac{1}{2} m v_1^2 \quad p_i = m v_1$

Final $\quad m v_1 = m v_f + m v_f \Rightarrow v_f = v_1/2$

$$K_f = \frac{1}{2}(m2)\left(\frac{1}{2} v_1\right)^2 = \frac{1}{4} m v_1^2$$

$$K_f - K_i = \frac{-1}{4} m v_1^2 \text{ goes to internal energy}$$

as heat, deformation, etc.

5. Energy and momentum conserved only if first train stops, second acquires v_1. This clearly satisfies conservation because have just changed labels effectively

6. a. $\sum \vec{F}_{ext} = 0 \Rightarrow 0 = \underbrace{N + T - m_1 g - m_2 g}_{y \text{ component}}$ and $\underbrace{f = 0}_{x \text{ comp.}}$

$\sum \vec{\tau}_{ext} = 0 = -m_2 g d \cos\theta - m_1 g \frac{L}{2} \cos\theta + TL\cos\theta$

$f = 0$, $T + N = (m_1 + m_2) g$, $TL = m_1 g \frac{L}{2} + m_2 g d$

b) $T = \frac{1}{2} m_1 g + m_2 g \frac{d}{L}$

c) $f = 0$ $\quad N = \frac{1}{2} m_1 g + m_2 g \left(1 - \frac{d}{L}\right)$

7. a) Net Torque vanishes when $m_1 g R_1 = m_2 g R_2$

$$\frac{m_1}{m_2} = \frac{R_2}{R_1} = \frac{1}{3}$$

b) Call a_1 and a_2 accelerations of m_1 and m_2 in y direction

$a_1 = \alpha R_1$ and $a_2 = -\alpha R_2$

$m_1 a_1 = -m_1 g + T_1 \qquad m_2 a_2 = -m_2 g + T_2$

$m_1 g + m_1 \alpha R_1 = T_1 \qquad m_2 g + m_2 \alpha R_2 = T_2$

Must also have net torque on wheel vanish since no inertia. (Otherwise infinite α)

$$T_1 R_1 = T_2 R_2$$

$R_1 m_1 g + m_1 \alpha R_1^2 = m_2 g R_2 + m_2 \alpha R_2^2$

$$\alpha = g(m_2 R_2 - m_1 R_1)/(m_1 R_1^2 + m_2 R_2^2)$$

7 c) $T_1 = m_1 g + m_1 R_1 \cdot g \dfrac{m_2 R_2 - m_1 R_1}{m_1 R_1^2 + m_2 R_2^2}$

$$T_2 = m_2 g + m_2 g \frac{m_2 R_2^2 - m_1 R_1 R_2}{m_1 R_1^2 + m_2 R_2^2}$$

8) a) Can treat both shells as point masses

$$\vec{F} = -\frac{G(M_1+M_2)m}{a^2}\hat{}$$

unit vector out radially

b) Outer shell gives no force

$$\vec{F} = -\frac{G M_1 m}{b^2}\hat{b}$$

c) $\vec{F} = 0$ since inside both shells.

d) Potential $= -\dfrac{G(M_1+M_2)m}{a}$

Must have kinetic energy equal to this in magnitude

$$\frac{1}{2}\cancel{m}v^2 = +G\frac{(M_1+M_2)\cancel{m}}{a}$$

$$v = \sqrt{\frac{2G(M_1+M_2)}{a}}$$

9) a) Fluid conservation says flow through 1 must equal that through 2

$$v_1 A_1 = v_2 A_2 \Rightarrow \frac{v_1}{v_2} = \frac{A_2}{A_1}$$

b) Bernoulli's equation tells us

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$ (no gravity since horizontal)

$$P_1 - P_2 = \frac{1}{2}\rho\left(v_2^2 - v_1^2\right)$$

$$= \frac{1}{2}\rho v_1^2\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right)$$

c) $$P_1 - P_2 = \rho' g h$$

$$h = \frac{1}{2}\frac{\rho v_1^2}{\rho' g}\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right)$$

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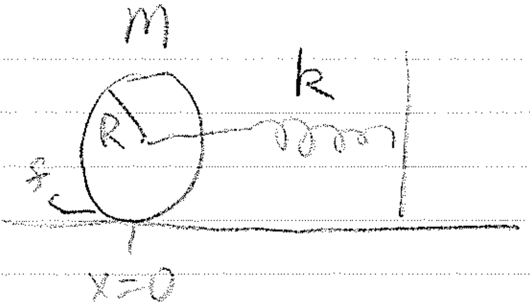


Normal forces cancel so ignore

friction on bottom

a) $$ma = -kx - f$$

$$\tau = I\alpha$$

$$-fR = mR^2\alpha \qquad f = -mR\alpha$$

But $\alpha = -\frac{a}{R}$ so $f = ma$

$$2ma = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{2m}x \qquad \omega^2 = \frac{k}{2m}$$

b) $$x = d\cos\omega t$$

c) Maximum kinetic energy when potential is zero $\Rightarrow$ $x=0$

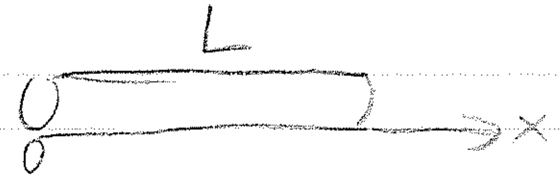
Reaches this point when $\omega t = \frac{\pi}{2}$, $t = \frac{\pi}{2\omega}$

$$K_{trans} = \frac{1}{2}mv^2 = \frac{1}{2}m(d\omega\sin\omega t)^2 = \frac{1}{2}md^2\omega^2$$

$$= \frac{1}{2}\cdot\frac{k}{2}d^2$$

d) At same time. Since $K_{trans} + K_{rot} = \frac{1}{2}kd^2$ at this point $K_{rot} = K_{trans} = \frac{1}{4}kd^2$

11) a



Both open $\rightarrow$ $p = 0$. Solved by $p_m \sin kx$
if $\sin kL = 0$

Lowest modes
$$kL = \pi$$
$$kL = 2\pi$$
$$kL = 3\pi$$

Frequency $f = \frac{\omega}{2\pi} = \frac{v}{\lambda}$ $\quad \lambda = \frac{2\pi}{k}$

$$f = \frac{vk}{2\pi}$$

lowest modes $\quad f = \frac{v}{2L}, \frac{v}{L}, \frac{3v}{2L}$

b) One closed, p is maximum at L

$\sin kL = \pm 1$ $\quad kL = \frac{\pi}{2}$
$$kL = \frac{3\pi}{2}$$
$$kL = \frac{5\pi}{2}$$

$$f = \frac{v}{4L}, \frac{3v}{4L}, \frac{5v}{4L}$$

c) Lowest frequency from tube closed at one end $\quad f = \frac{v}{4L}$

This mode has $L = \frac{1}{4}$ of wavelength $= \frac{17m}{4} = 4.25m$

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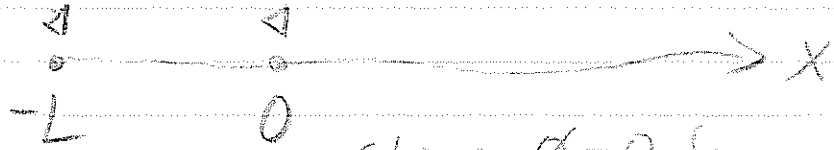

choose $\phi = 0$ for wave from origin

a) $\rho = \rho_m \sin(kx - \omega t) + \rho_m \sin(k(x+L) - \omega t)$

same phase at $x = -L$

$$= \rho_m \left[\sin(kx - \omega t) + \sin(kx - \omega t + kL)\right]$$

$$= \rho_m \left[\sin\left(kx - \omega t + \frac{kL}{2} - \frac{kL}{2}\right) + \sin\left(kx - \omega t + \frac{kL}{2} + \frac{kL}{2}\right)\right]$$

$$= \rho_m \sin\left(kx - \omega t + \frac{kL}{2}\right) \cos\frac{kL}{2}$$

here $v = \frac{\omega}{k}$

$$\rho = \rho_m \sin\left(\frac{\omega}{v} x - \omega t + \frac{\omega L}{2v}\right) \cos\frac{L\omega}{2v}$$

b) Energy must be conserved so ρ_m must drop off with distance.

c) Complete cancellation $\cos\frac{L\omega}{2v} = 0$

$$\frac{L\omega}{2v} = \left(n + \frac{1}{2}\right)\pi$$

$$\omega = \left(n + \frac{1}{2}\right)\frac{2\pi v}{L}$$

$$\frac{L}{\lambda} = n + \frac{1}{2}$$

d) $\frac{L\omega}{2v} = n\pi \qquad \omega = \frac{2\pi v}{L} n, \; L = n\lambda$