Classical Physics 171.105
Fall 2002, Second Mid-term Solutions

1) a) The gravitational force must provide the centripetal acceleration

$$|\vec{F}| = \frac{GMm}{R^2} = m|\vec{a}| = m\frac{v^2}{R}$$

$$v^2 = \frac{GM}{R} \qquad v = \sqrt{\frac{GM}{R}}$$

b) $$K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{R}$$

$$U = -\frac{GMm}{R} \qquad \frac{K}{U} = -\frac{1}{2}$$

c) The mechanical energy $U+K$ must decrease as energy goes into internal degrees of freedom.

$$U+K = -\frac{1}{2}\frac{GMm}{R}$$ will decrease as R decreases

$\rightarrow$ R decreases with time

2 a. The initial kinetic energy is

For $t<0$ $K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$ $\qquad I = \frac{1}{2}MR^2$

$\omega = \frac{v}{R}$ for rolling without slipping

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}\cdot\frac{1}{2}MR^2\cdot\left(\frac{v}{R}\right)^2$$
$$= \frac{3}{4}Mv^2$$

At highest point $K=0$ $\quad U = Mgh$

$$\frac{3}{4}Mv^2 = Mgh \Rightarrow h = \frac{3v^2}{4g}$$

b. The angular acceleration has magnitude $\alpha = \frac{a}{R}$ since $\omega = \frac{v}{R}$

We know the forces and hence accelerations are constant so the distance moved

$$\frac{h}{\sin\theta} = \frac{1}{2}at^2 \qquad t = \sqrt{\frac{2h}{a\sin\theta}}$$

Also know $-at = v$ $\qquad t^2 = \frac{v^2}{a^2}$

$$\frac{h}{\sin\theta} = \frac{1}{2}a\cdot\frac{v^2}{a^2} \qquad a = \frac{v^2\sin\theta}{2h} = \alpha R$$

$$\alpha = \frac{v^2\sin\theta}{2hR} = \frac{\cancel{v^2}\sin\theta}{2R\cdot 3\cancel{v^2}/\cancel{g}4} = \frac{2g\sin\theta}{3R}$$

2b. alternate

$$K = \frac{3M}{4} R^2 \omega^2 \qquad \frac{\partial K}{\partial t} = \frac{3}{2} MR^2 \omega\, \alpha = \frac{3}{2} MRv\alpha$$

$$\frac{\partial U}{\partial t} = Mg\frac{dh}{dt} = Mg\, v\sin\theta$$

$$\frac{\partial K}{\partial t} = -\frac{\partial U}{\partial t} \qquad -\frac{3}{2} MRv\alpha = Mg\, v\sin\theta$$

$$\alpha = \frac{2\, g\sin\theta}{3\, R}$$

3. Use the facts that

$$M\vec{a}_{cm} = \vec{F}_{ext} \quad \text{and} \quad \frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$$

Since impact is so brief that change in position and rotation can be ignored

$$M\vec{v}^{\,f}_{cm} = \int_0^{\Delta t} dt\, \vec{F}_{ext} = \vec{F}_{av} \cdot \Delta t$$

$$= F_{av} \Delta t\, \hat{j}$$ where $\hat{j}$ is unit vector along y axis

$$\vec{L}_f = \int_0^{\Delta t} dt\, \vec{\tau}_{ext} = \int_0^{\Delta t} dt \cdot \left(-\tfrac{L}{2}\hat{i}\right) \times F_{ext}\,\hat{j}$$

position

$$= -\frac{L}{2} F_{av} \Delta t\, \hat{k}$$ where $\hat{k}$ is unit vector out of plane

⇒ After collision center of mass moves up along y axis with speed $v = \frac{F_{av}\Delta t}{M}$

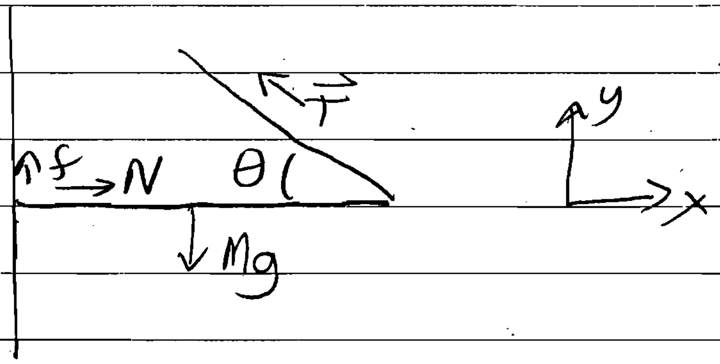
Rotates with angular Momentum $|\vec{L}| = \frac{L}{2} F_{av} \Delta t$ clockwise.

$$L = I\omega = \frac{1}{12} M L^2 \omega$$

$$\frac{1}{12} M L^2 \omega = \frac{L}{2} F_{av} \Delta t$$

$$\omega = \frac{6 F_{av} \Delta t}{M L}$$

4 a



In equilibrium $\quad \sum \vec{F}_{ext} = 0 \quad , \quad \sum \vec{\tau}_{ext} = 0$

Force along x $\quad N - T\cos\theta = 0$

" " y $\quad f + T\sin\theta - Mg = 0$

$N = T\cos\theta$

$f = Mg - T\sin\theta$

Choose to find torques about end of beam at wall

$$0 = -Mg \cdot \frac{L}{2} + T\sin\theta \cdot L \quad \Rightarrow \quad T\sin\theta = \frac{1}{2}Mg$$

so

$$f = \frac{1}{2}Mg \quad , \quad N = \frac{Mg}{2\tan\theta} \quad , \quad T = \frac{Mg}{2\sin\theta}$$

b) $\quad f < \mu N \quad \Rightarrow \quad \frac{1}{2}Mg < \mu \cdot \frac{Mg}{2\tan\theta}$

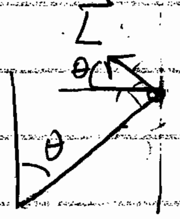
$$\tan\theta < \mu$$

5a) $z = L\cos\theta$

$x = L\sin\theta\cos\omega t$

$y = L\sin\theta\sin\omega t$

b) $\vec{L} = \vec{r} \times M\vec{v}$



$$\left.\begin{aligned} v_x &= -L\omega\sin\theta\sin\omega t \\ v_y &= L\omega\sin\theta\cos\omega t \end{aligned}\right) \vec{v} \perp \text{to } \vec{r}$$

$\vec{L}$ must be perpendicular to both $\vec{r}$ and $\vec{v}$

Since $\vec{v} \perp$ to $\vec{r}$ $\quad |\vec{L}| = Mvr$

$= ML^2\omega\sin\theta$

Directed at angle θ above horizontal and towards center (opposite of $\vec{r}$)

$L_z = mL^2\omega\sin^2\theta$

$L_x = -ML^2\omega\sin\theta\cos\theta\cos\omega t$

$L_y = -ML^2\omega\sin\theta\cos\theta\sin\omega t$

c) No. Reason is that rotation is not about principal axis

d) $\vec{\tau} = \frac{d\vec{L}}{dt}$ $\quad \tau_x = ML^2\omega^2\sin\theta\cos\theta\sin\omega t$

$\tau_y = ML^2\omega^2\sin\theta\cos\theta\cos\omega t$