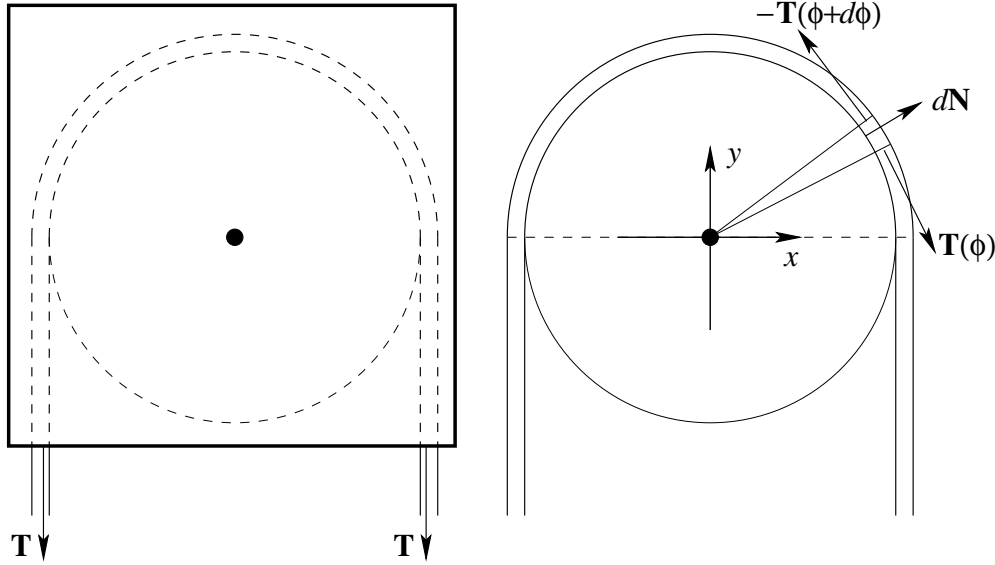


## String tension and the force acting on a pulley

In class I pointed out that a string wrapped around a pulley applies an overall downward force of magnitude  $2T$  to the pulley, where  $T$  is the string tension. This was justified indirectly, by treating the pulley and the middle of the string as a black box (left panel of the figure). Let us justify this result by a direct calculation.



Consider an element of the string between the angles  $\phi$  and  $\phi + d\phi$  measured from the horizontal direction (right panel). It is pulled by the adjacent segment on its right with the force  $\mathbf{T}(\phi)$ . In its turn, it exerts the force  $\mathbf{T}(\phi + d\phi)$  on the next segment to its left; the segment on the left exerts on it an equal and opposite force  $-\mathbf{T}(\phi + d\phi)$ . Thus the net force applied to the segment by its neighbors is  $-\mathbf{T}(\phi + d\phi) + \mathbf{T}(\phi)$ . Componentwise,

$$\mathbf{T}(\phi) = (-T \sin \phi, T \cos \phi, 0), \quad \mathbf{T}(\phi + d\phi) = (-T \sin(\phi + d\phi), T \cos(\phi + d\phi), 0). \quad (1)$$

Although  $-\mathbf{T}(\phi + d\phi)$  and  $\mathbf{T}(\phi)$  are equal in magnitude ( $T$ ), they are slightly misaligned and therefore do not cancel out exactly:

$$\mathbf{T}(\phi) - \mathbf{T}(\phi + d\phi) = (-T \cos \phi d\phi, -T \sin \phi d\phi, 0). \quad (2)$$

Because the segment is stationary, the net force acting on it must be zero. To compensate for the imperfect cancellation of the misaligned tension forces, the pulley must exert a normal force (no friction!)  $d\mathbf{N}$  such that  $\mathbf{T}(\phi) - \mathbf{T}(\phi + d\phi) + d\mathbf{N} = 0$ , whence

$$d\mathbf{N} = (T \cos \phi d\phi, T \sin \phi d\phi, 0). \quad (3)$$

(Verify that  $d\mathbf{N}$  is indeed normal to the surface.)

By Newton's 3rd law, the segment exerts an equal and opposite force

$$d\mathbf{F} = (-T \cos \phi d\phi, -T \sin \phi d\phi, 0) \quad (4)$$

on the pulley. We can now integrate the contributions of all segments between  $\phi = 0$  and  $\pi$  to obtain the net force acting on the pulley:

$$\mathbf{F} = \int_0^\pi \frac{d\mathbf{F}}{d\phi} d\phi = \left( -\int_0^\pi T \cos \phi d\phi, -\int_0^\pi T \sin \phi d\phi, 0 \right) = (0, -2T, 0). \quad (5)$$

Thus, indeed, the net force by the string on the pulley has the magnitude  $2T$  and points downward.