Final Exam, Introduction to Quantum Mechanics

December 17, 2003

Work all problems. Put each problem on a separate page, and be sure to put your name on all the pages. Points are assigned as indicated; it is in your interest to look over the whole exam and work what seem to you to be the easiest parts first. Don't get too hung up on any one problem and waste all your time on that one, thereby neglecting the rest of the exam.

Very Important Note: In many of these problems you are given actual *numbers.* You *must* plug these numbers in to receive full credit. Please don't cost yourself easy points by forgetting to do this. Also, don't set $\hbar = 1$ unless I tell you explicitly in that problem that you can.

Good Luck!!

Possibly useful operators

Simple Harmonic Oscillator:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\,\hat{p}\right)$$

Spin-$\frac{1}{2}$:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spin-1:

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Possibly useful integrals

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^{2n}\, e^{-ax^2} = \begin{cases} \sqrt{\pi/a} & n=0 \\ \frac{1}{2}\sqrt{\pi/a^3} & n=1 \\ \frac{3}{4}\sqrt{\pi/a^5} & n=2 \end{cases}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2+bx} = \sqrt{\pi/a}\cdot e^{b^2/4a}$$

$$\int_{0}^{\infty} \mathrm{d}x\, x^{n}\, e^{-ax} = \frac{n!}{a^{n+1}}$$

$$\int \mathrm{d}x\ \sin ax \sin bx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}$$

$$\int \mathrm{d}x\ \cos ax \cos bx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}$$

$$\int \mathrm{d}x\ \sin ax \cos bx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}$$

$$\int \mathrm{d}x\ \sin ax \cos ax = -\frac{\cos^2 ax}{2a}$$

$$\int \mathrm{d}x\ \sin^2 ax = \frac{x}{2} - \frac{\sin 2ax}{4a} \qquad \int \mathrm{d}x\ \cos^2 ax = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \mathrm{d}x\, x \sin ax \sin bx = \frac{\cos(a-b)x}{2(a-b)^2} - \frac{\cos(a+b)x}{2(a+b)^2} + x\frac{\sin(a-b)x}{2(a-b)} - x\frac{\sin(a+b)x}{2(a+b)}$$

$$\int \mathrm{d}x\, x \cos ax \cos bx = \frac{\cos(a-b)x}{2(a-b)^2} + \frac{\cos(a+b)x}{2(a+b)^2} + x\frac{\sin(a-b)x}{2(a-b)} + x\frac{\sin(a+b)x}{2(a+b)}$$

$$\int \mathrm{d}x\, x \sin ax \cos bx = \frac{\sin(a-b)x}{2(a-b)^2} + \frac{\sin(a+b)x}{2(a+b)^2} - x\frac{\cos(a-b)x}{2(a-b)} - x\frac{\cos(a+b)x}{2(a+b)}$$

$$\int \mathrm{d}x\, x \sin ax \cos ax = \frac{\sin 2ax}{8a^2} - x\frac{\cos 2ax}{4a}$$

$$\int \mathrm{d}x\, x \sin^2 ax = \frac{x^2}{4} - \frac{\cos 2ax}{8a^2} - \frac{x \sin 2ax}{4a}$$

$$\int \mathrm{d}x\, x \cos^2 ax = \frac{x^2}{4} + \frac{\cos 2ax}{8a^2} + \frac{x \sin 2ax}{4a}$$

1. (25 pts) A coherent state $|\alpha\rangle$ of the simple harmonic oscillator is one that satisfies the equation

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\,,$$

where α is a complex number, $\alpha = |\alpha|e^{i\varphi}$.

a. (5 pts.) Compute $\langle\alpha|\hat{N}^2|\alpha\rangle$, where $\hat{N}$ is the number operator.

b. (20 pts.) Show that Δx in the state $|\alpha\rangle$ is independent of α.

2. (20 pts.) Consider three particles, with spin quantum numbers $s_1 = \frac{1}{2}$, $s_2 = \frac{3}{2}$ and $s_3 = 1$.

a. (5 pts.) What is the total number of possible states $|s_1 s_2 s_3; m_1 m_2 m_3\rangle$?

b. (5 pts.) Define $\hat{S}_{12} = \hat{S}_1 + \hat{S}_2$, with eigenstates $|s_{12} m_{12}\rangle$. What is $|s_{12} m_{12}\rangle = |11\rangle$ in terms of the states $|s_1 m_1\rangle$ and $|s_2 m_2\rangle$?

c. (10 pts.) Define the total spin $\hat{S} = \hat{S}_1 + \hat{S}_2 + \hat{S}_3$, with total spin quantum number s. What are the possible values of s? Are there any of these values which can be made in more than one way? Verify that the total number of states $|sm\rangle$ is the same as in part a.

3. (40 pts.) Consider the potential

$$V(x) = \begin{cases} 0 & x < -\dfrac{\pi}{2} \\ -V_0 & -\dfrac{\pi}{2} \le x \le 0 \\ \infty & x > 0 \end{cases}$$

In this problem you may set $\hbar = 1$.

a. (10 pts.) Set $V_0 = 5$ (so $-V_0 = -5$ in the potential). A plane wave of particles with mass $m = 2$ and energy $E = 4$ is incident from $x = -\infty$. Write the general solution to the Schrödinger equation in the regions $-\frac{\pi}{2} \le x \le 0$ and $x < -\frac{\pi}{2}$. Use all of the numerical information that is provided.

b. (10 pts.) What are the boundary conditions that must be imposed at $x = 0$, $x = -\frac{\pi}{2}$ and $x = -\infty$?

c. (10 pts.) Find the reflection coefficient R.

d. (5 pts.) Now consider solutions for which $E < 0$. What is the general solution to the Schrödinger equation in the two regions, after the boundary conditions at $x = 0$ and $x = -\infty$ are taken into account?

e. (5 pts.) If you imposed the boundary conditions, you would find a transcendental equation for E which you would have to solve numerically. For general V_0, will this equation *always* have at least one solution? Why or why not? (No credit for just guessing correctly!)

4. (30 pts.) Suppose we have a particle of mass $m = 3$ in a one-dimensional infinite square well with walls at $x = -\pi$ and $x = \pi$. At $t = 0$, the wavefunction of the particle is given by

$$\Psi(x, 0) = \sqrt{\frac{1}{3\pi}} \cos \frac{x}{2} + i\sqrt{\frac{2}{3\pi}} \sin x \, .$$

In this problem you may set $\hbar = 1$.

a. (10 pts.) Compute $\Psi(x, t)$ for $t > 0$.

b. (5 pts.) What is $\langle H \rangle$, the expectation value of the energy?

c. (15 pts.) Compute $\langle x \rangle$ for general $t > 0$.

5. (35 pts.) Consider a spin-$\frac{1}{2}$ particle described by the Hamiltonian

$$\hat{H} = \omega_1 \hat{S}_x + \omega_2 \hat{S}_z \, ,$$

where $\omega_1 = 3$ and $\omega_2 = 4$. In this problem you may set $\hbar = 1$.

a. (5 pts.) What is the representation of $\hat{H}$ in the basis in which $\hat{S}_z$ is diagonal?

b. (15 pts.) Find the stationary states and the corresponding energy eigenvalues.

c. (15 pts.) Suppose that at $t = 0$ the particle is in an state in which $S_z = \frac{1}{2}\hbar$. What is the probability of finding $S_z = -\frac{1}{2}\hbar$ for $t > 0$?

6. (35 pts.) This problem concerns a particle of spin 1. The particle is initially in a state $|\psi\rangle$ which is represented in the S_z basis by the column vector

$$A \begin{pmatrix} i \\ 3 \\ -1 \end{pmatrix}$$

a. (5 pts.) Use the normalization of $|\psi\rangle$ to find A.

b. (10 pts.) Compute $\langle S_y \rangle$ in the state $|\psi\rangle$.

c. (15 pts.) What is the operator $\hat{R}(\alpha)$ that rotates a general state $|\psi\rangle$ by an angle α about the $\hat{z}$ axis? Find the matrix representing $\hat{R}(\pi/2)$ in the S_z basis.

d. (5 pts.) Compute $|\varphi\rangle = \hat{R}(\pi/2)|\psi\rangle$, and represent the state $|\varphi\rangle$ in the S_z basis.

7. (15 pts.) Consider three Hermitian operators, A, B and C. Suppose that

$$[B, C] = A \qquad \text{and} \qquad [A, C] = B \, .$$

Show that in any state,

$$\Delta(AB) \cdot \Delta C \geq \frac{1}{2}\langle aA^2 + bB^2 + cC^2 \rangle \, ,$$

and find the constants a, b and c.