Final Exam, Introduction to Quantum Mechanics

December 15, 2004

Work all problems. Put each problem on a separate page, and be sure to put your name on all the pages. Points are assigned as indicated; it is in your interest to look over the whole exam and work what seem to you to be the easiest parts first. Don't get too hung up on any one problem and waste all your time on that one, thereby neglecting the rest of the exam.

Very Important Note: In many of these problems you are given actual *numbers*. You *must* plug these numbers in to receive full credit. Please don't cost yourself easy points by forgetting to do this. Also, don't set $\hbar = 1$ unless I tell you explicitly in that problem that you can.

Good Luck!!

Possibly useful operators

Simple Harmonic Oscillator:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right)$$

Spin-$\frac{1}{2}$:

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spin-1:

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Possibly useful integrals

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^{2n}\, e^{-ax^2} = \begin{cases} \sqrt{\pi/a} & n=0 \\ \frac{1}{2}\sqrt{\pi/a^3} & n=1 \\ \frac{3}{4}\sqrt{\pi/a^5} & n=2 \end{cases}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2+bx} = \sqrt{\pi/a}\cdot e^{b^2/4a} \qquad \int_0^{\infty} \mathrm{d}x\, x^n\, e^{-ax} = \frac{n!}{a^{n+1}}$$

$$\int \mathrm{d}x\ \sin ax \sin bx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}$$

$$\int \mathrm{d}x\ \cos ax \cos bx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}$$

$$\int \mathrm{d}x\ \sin ax \cos bx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}$$

$$\int \mathrm{d}x\ \sin ax \cos ax = -\frac{\cos^2 ax}{2a}$$

$$\int \mathrm{d}x\ \sin^2 ax = \frac{x}{2} - \frac{\sin 2ax}{4a} \qquad \int \mathrm{d}x\ \cos^2 ax = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \mathrm{d}x\, x \sin ax \sin bx = \frac{\cos(a-b)x}{2(a-b)^2} - \frac{\cos(a+b)x}{2(a+b)^2} + x\frac{\sin(a-b)x}{2(a-b)} - x\frac{\sin(a+b)x}{2(a+b)}$$

$$\int \mathrm{d}x\, x \cos ax \cos bx = \frac{\cos(a-b)x}{2(a-b)^2} + \frac{\cos(a+b)x}{2(a+b)^2} + x\frac{\sin(a-b)x}{2(a-b)} + x\frac{\sin(a+b)x}{2(a+b)}$$

$$\int \mathrm{d}x\, x \sin ax \cos bx = \frac{\sin(a-b)x}{2(a-b)^2} + \frac{\sin(a+b)x}{2(a+b)^2} - x\frac{\cos(a-b)x}{2(a-b)} - x\frac{\cos(a+b)x}{2(a+b)}$$

$$\int \mathrm{d}x\, x \sin ax \cos ax = \frac{\sin 2ax}{8a^2} - x\frac{\cos 2ax}{4a}$$

$$\int \mathrm{d}x\, x \sin^2 ax = \frac{x^2}{4} - \frac{\cos 2ax}{8a^2} - \frac{x \sin 2ax}{4a}$$

$$\int \mathrm{d}x\, x \cos^2 ax = \frac{x^2}{4} + \frac{\cos 2ax}{8a^2} + \frac{x \sin 2ax}{4a}$$

$$\int \mathrm{d}x\, x^2 \cos^2 ax = \frac{x^3}{6} + \frac{x\cos 2ax}{4a^2} + \frac{(2a^2x^2-1)\sin 2ax}{8a^3}$$

$$\int \mathrm{d}x\, x^2 \sin^2 ax = \frac{x^3}{6} - \frac{x\cos 2ax}{4a^2} + \frac{(1-2a^2x^2)\sin 2ax}{8a^3}$$

$$\int \mathrm{d}x\, x^2 \sin ax \cos ax = \frac{x \sin 2ax}{4a^2} + + \frac{(1-2a^2x^2)\cos 2ax}{8a^3}$$

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over every coe cient, e.g., for −8/15 read −$\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
. .			

$Y_1^0 = \sqrt{\frac{3}{4}}\cos$

$Y_1^1 = -\sqrt{\frac{3}{8}}\sin e^{i}$

$Y_2^0 = \sqrt{\frac{5}{4}}\left(\frac{3}{2}\cos^2 - \frac{1}{2}\right)$

$Y_2^1 = -\sqrt{\frac{15}{8}}\sin \cos e^{i}$

$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2}}\sin^2 e^{2i}$

$Y^{-m} = (-1)^m Y^{m}$

$d_{m,0} = \sqrt{\frac{4}{2+1}}\, Y^m e^{-im}$

$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$

1/2×1/2

m_1 m_2	1 +1
+1/2 +1/2	1

m_1 m_2	1 0	0 0
+1/2 ×1/2	1/2	1/2
×1/2 +1/2	1/2	×1/2

m_1 m_2	1 ×1
×1/2 ×1/2	1

1×1/2

m_1 m_2	3/2 +3/2
+1 +1/2	1

m_1 m_2	3/2 +1/2	1/2 +1/2
+1 ×1/2	1/3	2/3
0 +1/2	2/3	×1/3

m_1 m_2	3/2 ×1/2	1/2 ×1/2
0 ×1/2	2/3	1/3
×1 +1/2	1/3	×2/3

m_1 m_2	3/2 ×3/2
×1 ×1/2	1

2×1/2

m_1 m_2	5/2 +5/2
+2 +1/2	1

m_1 m_2	5/2 +3/2	3/2 +3/2
+2 ×1/2	1/5	4/5
+1 +1/2	4/5	×1/5

m_1 m_2	5/2 +1/2	3/2 +1/2
+1 ×1/2	2/5	3/5
0 +1/2	3/5	×2/5

m_1 m_2	5/2 ×1/2	3/2 ×1/2
0 ×1/2	3/5	2/5
×1 +1/2	2/5	×3/5

m_1 m_2	5/2 ×3/2	3/2 ×3/2
×1 ×1/2	4/5	1/5
×2 +1/2	1/5	×4/5

m_1 m_2	5/2 ×5/2
×2 ×1/2	1

3/2×1/2

m_1 m_2	2 +2
+3/2 +1/2	1

m_1 m_2	2 +1	1 +1
+3/2 ×1/2	1/4	3/4
+1/2 +1/2	3/4	×1/4

m_1 m_2	2 0	1 0
+1/2 ×1/2	1/2	1/2
×1/2 +1/2	1/2	×1/2

m_1 m_2	2 ×1	1 ×1
×1/2 ×1/2	3/4	1/4
×3/2 +1/2	1/4	×3/4

m_1 m_2	2 ×2
×3/2 ×1/2	1

2×1

m_1 m_2	3 +3
+2 +1	1

m_1 m_2	3 +2	2 +2
+2 0	1/3	2/3
+1 +1	2/3	×1/3

m_1 m_2	3 +1	2 +1	1 +1
+2 ×1	1/15	1/3	3/5
+1 0	8/15	1/6	×3/10
0 +1	2/5	×1/2	1/10

m_1 m_2	3 0	2 0	1 0
+1 ×1	1/5	1/2	3/10
0 0	3/5	0	×2/5
×1 +1	1/5	×1/2	3/10

m_1 m_2	3 ×1	2 ×1	1 ×1
0 ×1	2/5	1/2	1/10
×1 0	8/15	×1/6	×3/10
×2 +1	1/15	×1/3	3/5

m_1 m_2	3 ×2	2 ×2
×1 ×1	2/3	1/3
×2 0	1/3	×2/3

m_1 m_2	3 ×3
×2 ×1	1

1×1

m_1 m_2	2 +2
+1 +1	1

m_1 m_2	2 +1	1 +1
+1 0	1/2	1/2
0 +1	1/2	×1/2

m_1 m_2	2 0	1 0	0 0
+1 ×1	1/6	1/2	1/3
0 0	2/3	0	×1/3
×1 +1	1/6	×1/2	1/3

m_1 m_2	2 ×1	1 ×1
0 ×1	1/2	1/2
×1 0	1/2	×1/2

m_1 m_2	2 ×2
×1 ×1	1

3/2×1

m_1 m_2	5/2 +5/2
+3/2 +1	1

m_1 m_2	5/2 +3/2	3/2 +3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	×2/5

m_1 m_2	5/2 +1/2	3/2 +1/2	1/2 +1/2
+3/2 ×1	1/10	2/5	1/2
+1/2 0	3/5	1/15	×1/3
×1/2 +1	3/10	×8/15	1/6

m_1 m_2	5/2 ×1/2	3/2 ×1/2	1/2 ×1/2
+1/2 ×1	3/10	8/15	1/6
×1/2 0	3/5	×1/15	×1/3
×3/2 +1	1/10	×2/5	1/2

m_1 m_2	5/2 ×3/2	3/2 ×3/2
×1/2 ×1	3/5	2/5
×3/2 0	2/5	×3/5

m_1 m_2	5/2 ×5/2
×3/2 ×1	1

$d^j_{m\,m} = (-1)^{m-m} d^j_{m\,m} = d^j_{-m,-m}$

3/2×3/2

m_1 m_2	3 +3
+3/2 +3/2	1

m_1 m_2	3 +2	2 +2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	×1/2

m_1 m_2	3 +1	2 +1	1 +1
+3/2 ×1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	×2/5
×1/2 +3/2	1/5	×1/2	3/10

m_1 m_2	3 0	2 0	1 0	0 0
+3/2 ×3/2	1/20	1/4	9/20	1/4
+1/2 ×1/2	9/20	1/4	×1/20	×1/4
×1/2 +1/2	9/20	×1/4	×1/20	1/4
×3/2 +3/2	1/20	×1/4	9/20	×1/4

m_1 m_2	3 ×1	2 ×1	1 ×1
+1/2 ×3/2	1/5	1/2	3/10
×1/2 ×1/2	3/5	0	×2/5
×3/2 +1/2	1/5	×1/2	3/10

m_1 m_2	3 ×2	2 ×2
×1/2 ×3/2	1/2	1/2
×3/2 ×1/2	1/2	×1/2

m_1 m_2	3 ×3
×3/2 ×3/2	1

2×3/2

m_1 m_2	7/2 +7/2
+2 +3/2	1

m_1 m_2	7/2 +5/2	5/2 +5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	×3/7

m_1 m_2	7/2 +3/2	5/2 +3/2	3/2 +3/2
+2 ×1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	×2/5
0 +3/2	2/7	×18/35	1/5

m_1 m_2	7/2 +1/2	5/2 +1/2	3/2 +1/2	1/2 +1/2
+2 ×3/2	1/35	6/35	2/5	2/5
+1 ×1/2	12/35	5/14	0	×3/10
0 +1/2	18/35	×3/35	×1/5	1/5
×1 +3/2	4/35	×27/70	2/5	×1/10

m_1 m_2	7/2 ×1/2	5/2 ×1/2	3/2 ×1/2	1/2 ×1/2
+1 ×3/2	4/35	27/70	2/5	1/10
0 ×1/2	18/35	3/35	×1/5	×1/5
×1 +1/2	12/35	×5/14	0	3/10
×2 +3/2	1/35	×6/35	2/5	×2/5

m_1 m_2	7/2 ×3/2	5/2 ×3/2	3/2 ×3/2
0 ×3/2	2/7	18/35	1/5
×1 ×1/2	4/7	×1/35	×2/5
×2 +1/2	1/7	×16/35	2/5

m_1 m_2	7/2 ×5/2	5/2 ×5/2
×1 ×3/2	4/7	3/7
×2 ×1/2	3/7	×4/7

m_1 m_2	7/2 ×7/2
×2 ×3/2	1

2×2

m_1 m_2	4 +4
+2 +2	1

m_1 m_2	4 +3	3 +3
+2 +1	1/2	1/2
+1 +2	1/2	×1/2

m_1 m_2	4 +2	3 +2	2 +2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	×3/7
0 +2	3/14	×1/2	2/7

m_1 m_2	4 +1	3 +1	2 +1	1 +1
+2 ×1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	×1/14	×3/10
0 +1	3/7	×1/5	×1/14	3/10
×1 +2	1/14	×3/10	3/7	×1/5

m_1 m_2	4 0	3 0	2 0	1 0	0 0
+2 ×2	1/70	1/10	2/7	2/5	1/5
+1 ×1	8/35	2/5	1/14	×1/10	×1/5
0 0	18/35	0	×2/7	0	1/5
×1 +1	8/35	×2/5	1/14	1/10	×1/5
×2 +2	1/70	×1/10	2/7	×2/5	1/5

m_1 m_2	4 ×1	3 ×1	2 ×1	1 ×1
+1 ×2	1/14	3/10	3/7	1/5
0 ×1	3/7	1/5	×1/14	×3/10
×1 0	3/7	×1/5	×1/14	3/10
×2 +1	1/14	×3/10	3/7	×1/5

m_1 m_2	4 ×2	3 ×2	2 ×2
0 ×2	3/14	1/2	2/7
×1 ×1	4/7	0	×3/7
×2 0	3/14	×1/2	2/7

m_1 m_2	4 ×3	3 ×3
×1 ×2	1/2	1/2
×2 ×1	1/2	×1/2

m_1 m_2	4 ×4
×2 ×2	1

$d^1_{0,0} = \cos$

$d^{1/2}_{1/2,1/2} = \cos\frac{\ }{2}$

$d^{1/2}_{1/2,-1/2} = -\sin\frac{\ }{2}$

$d^1_{1,1} = \frac{1+\cos}{2}$

$d^1_{1,0} = -\frac{\sin}{\sqrt{2}}$

$d^1_{1,-1} = \frac{1-\cos}{2}$

$d^{3/2}_{3/2,3/2} = \frac{1+\cos}{2}\cos\frac{\ }{2}$

$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos}{2}\sin\frac{\ }{2}$

$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos}{2}\cos\frac{\ }{2}$

$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos}{2}\sin\frac{\ }{2}$

$d^{3/2}_{1/2,1/2} = \frac{3\cos - 1}{2}\cos\frac{\ }{2}$

$d^{3/2}_{1/2,-1/2} = -\frac{3\cos + 1}{2}\sin\frac{\ }{2}$

$d^2_{2,2} = \left(\frac{1+\cos}{2}\right)^2$

$d^2_{2,1} = -\frac{1+\cos}{2}\sin$

$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2$

$d^2_{2,-1} = -\frac{1-\cos}{2}\sin$

$d^2_{2,-2} = \left(\frac{1-\cos}{2}\right)^2$

$d^2_{1,1} = \frac{1+\cos}{2}(2\cos - 1)$

$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin \cos$

$d^2_{1,-1} = \frac{1-\cos}{2}(2\cos + 1)$

$d^2_{0,0} = \frac{3}{2}\cos^2 - \frac{1}{2}$

Figure 34.1: The sign convention is that of Wigner (Group Theory, Academic Press, New York, 1959), also used by Condon and Shortley (The Theory of Atomic Spectra, Cambridge Univ. Press, New York, 1953), Rose (Elementary Theory of Angular Momentum, Wiley, New York, 1957), and Cohen (Tables of the Clebsch-Gordan Coe cients, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coe cients here have been calculated using computer programs written independently by Cohen and at LBNL.

1. (15 pts.) Consider three particles, with spin quantum numbers $s_1 = \frac{1}{2}$, $s_2 = 1$ and $s_3 = \frac{3}{2}$.

a. (5 pts.) What is the total number of states $|s_1 s_2 s_3; m_1 m_2 m_3\rangle$?

b. (5 pts.) Let the total spin be $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$. What are the possible eigenvalues of the operator S^2?

c. (5 pts.) If $m_1 = -\frac{1}{2}$, $m_2 = -1$ and $m_3 = -\frac{3}{2}$, what are the probabilites that s takes on each of its possible values?

2. (20 pts) The Hamiltonian for the simple harmonic oscillator is the sum of the kinetic and potential energies,

$$\hat{H} = T(\hat{p}_x) + V(\hat{x}) = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2}m\omega^2\hat{x}^2 .$$

Use raising and lowering operators to prove that in any energy eigenstate $|n\rangle$, we have $\langle T\rangle = \langle V\rangle$.

3. (15 pts.) Consider the operator $\hat{W} = \hat{x}^2\hat{p} + \hat{p}\hat{x}^2$.

a. (5 pts.) Prove that $\hat{W}$ corresponds to a physical observable.

b. (10 pts.) Derive a lower bound on $\Delta W \cdot \Delta p$ in a general state $|\psi\rangle$.

4. (30 pts.) Suppose we have a particle of mass $m = 3$ in a one-dimensional infinite square well with walls at $x = 0$ and $x = 2\pi$. The particle is in the ground state of the potential.

a. (5 pts.) What are the wavefunction $\psi(x)$ and energy E of this particle?

b. (10 pts.) Compute Δx for this state.

c. (5 pts.) Now suppose that the wall at $x = 2\pi$ is suddenly moved to $x = 4\pi$. What are the energy eigenvalues of the new potential?

d. (10 pts.) What is the probability that the particle is in the first excited state of the new potential?

5. (40 pts.) Suppose we have a spin-$\frac{1}{2}$ particle in the state $|\psi(0)\rangle$ at time $t = 0$. In the z-basis, , in which $\hat{S}_z$ is diagonal, $|\psi(0)\rangle$ is represented by

$$|\psi(0)\rangle = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

a. (15 pts.) Compute ΔS_x at $t = 0$.

b. (15 pts.) The Hamiltonian for this system is, in the z-basis,

$$H = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

Find the stationary states and the corresponding energy eigenvalues.

c. (10 pts.) Find $|\psi(t)\rangle$ for $t > 0$. In this part and the next you may set $\hbar = 1$.

d. (5 pts.) What is $\langle S_z \rangle$ at $t = \pi/5$?

6. (25 pts.) Consider the potential

$$V(x) = 2\delta(x) + 5\theta(x).$$

Recall that $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$. A plane wave of particles with mass $m = 2$ and energy $E = 9$ is incident from $x = -\infty$. In this problem you may set $\hbar = 1$.

a. (5 pts.) What is the general form of the solution to the time-independent Schrödinger equation in the two regions $x < 0$ and $x > 0$? How many arbitrary constants are there?

b. (10 pts.) What are the boundary conditions that must be imposed at $x = \pm\infty$ and $x = 0$? Apply them to your general solution.

c. (10 pts.) Find the transmission coefficient T.

7. (25 pts.) Consider a particle of mass $m = 4$ moving in the potential

$$V(x) = \begin{cases} \infty & x < 0 \\ 2x & x \geq 0 \end{cases}$$

Use the WKB approximation to estimate the ground state energy E_0. You may set $\hbar = 1$.

8. (30 pts.) This problem concerns a particle of spin 1.

a. (5 pts.) The particle is initially in a state $|\psi_1\rangle$ for which $\hat{S}_x|\psi_1\rangle = -\hbar|\psi_1\rangle$. What is the representation of $|\psi_1\rangle$ in the z-basis, in which $\hat{S}_z$ is diagonal?

b. (15 pts.) What is the operator $\hat{R}(\alpha)$ which rotates a general state $|\psi\rangle$ by an angle α about the $\hat{z}$ axis? Compute

$$|\psi_2\rangle = \hat{R}(\pi/2)|\psi_1\rangle,$$

and express the rotated state in the z-basis. (Hint: expand $|\psi_1\rangle$ in eigenstates of $\hat{R}(\alpha)$.) Show that $\hat{S}_y|\psi_2\rangle = -\hbar|\psi_2\rangle$.

c. (10 pts.) Compute the 3×3 matrix $\hat{R}(\pi)$ in the z-basis.