9/11 Vector Spaces - A Crash Course!

<u>Def</u>: A <u>VECTOR SPACE</u> V is a set of objects called <u>VECTORS</u> and a field F of <u>SCALARS</u> and two operations $(+,\cdot)$ such that:

i) V closed under $+$: $\vec{x}+\vec{y} \in V \quad \forall \vec{x},\vec{y} \in V$

ii) V closed under $\cdot$: $a\vec{x} \in V \quad \forall \vec{x} \in V,\ a \in F$

iii) $(V,+)$ ~~associative~~ commutative : $\vec{x}+\vec{y} = \vec{y}+\vec{x}$

iv) $(V,+)$ ~~commutative~~ associative : $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$

v) $\exists$ <u>ZERO</u> element : $\vec{x}+0 = \vec{x}$

vi) $\exists$ negatives : $\vec{x}+(-\vec{x}) = 0$

vii) Scalars are associative : $a(b\vec{x}) = (ab)\vec{x}$

viii) Scalars distribute : $a(\vec{x}+\vec{y}) = a\vec{x}+a\vec{y}$
$(a+b)\vec{x} = a\vec{x}+b\vec{x}$

ix) $\exists$ IDENTITY : $1\vec{x} = \vec{x}$

In this course, the field F will generally be $\mathbb{C}$. ("Complex Vector Space") but it could also be $\mathbb{R}$.

<u>Ex</u>: 1) $\mathbb{R}^n$ - The space of n-dimensional vectors $(F=\mathbb{R})$

2) C^n - the space of continuous (and n-differentiable) functions:

$$(f_1+f_2)(x) = f_1(x)+f_2(x) \in C^n \quad \checkmark$$
$$(af)(x) = af(x) \in C^n \quad \checkmark$$

3) Space of all infinite sequences $\{x_n\}_{n=1}^{\infty} \ni \sum_{n=1}^{\infty} |x_n|^2 < \infty$

Addition/Multiplication is the same as $\mathbb{R}^n$.

4) The ~~set~~ space of all fcns that are solutions of $\frac{d^2f}{dx^2} = -\omega^2 f$

$$f_1 = \sin \omega x$$
$$f_2 = \cos \omega x$$

5) L^2 - The space of all functions on $\mathbb{R} \ni \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$

Def: A set of vectors $\{\vec{x}_i\} \in V$ is said to <u>SPAN</u> the space V if <u>ANY</u> vector $\vec{v} \in V$ can be written as a linear combination of the $\{\vec{x}_i\}$, i.e.: $\vec{v} = \sum_i c_i \vec{x}_i$.

Def: A set of vectors $\{\vec{x}_i \in V\}$ is <u>LINEARLY INDEPENDENT</u> if:

$$\sum_i c_i \vec{x}_i = 0 \iff c_i = 0 \ \forall i$$

Def: A <u>BASIS OF V</u> is a set of vectors $\vec{x}_i \in V$ that are both linearly independent and span V. The number of vectors in a basis is called the <u>DIMENSION OF V</u>.

Ex: 1) In $\mathbb{R}^n$, there are n-linearly independent vectors that span $\mathbb{R}^n$, hence, $\mathbb{R}^n$ is n-dimensional.

Ex: $\{(0,1), (1,0)\}$ and $\{(1,0), (1,1)\}$ are bases for $\mathbb{R}^2$.

2) Example of sol$^{\underline{n}}$s of $\frac{d^2 f}{dx^2} = -\omega^2 f$: $f(x) = a \cos \omega x + b \sin \omega x$

$\Rightarrow$ Basis for this vector space is $\{\cos(\omega x), \sin(\omega x)\} \Rightarrow$ 2-dimensional.

Def: An <u>INNER PRODCT</u> takes 2 vectors and returns a scalar. It is written $\langle \vec{v} | \vec{w} \rangle$ in this course. The inner product obeys four axioms:

i) $\langle \vec{u} | \vec{v} + \vec{w} \rangle = \langle \vec{u} | \vec{v} \rangle + \langle \vec{u} | \vec{w} \rangle$

ii) $\langle \vec{u} | a\vec{v} \rangle = a \langle \vec{u} | \vec{v} \rangle$ ~~and~~

iii) $\langle \vec{u} | \vec{v} \rangle = \langle \vec{v} | \vec{u} \rangle^*$ ("Hermiticity")

iv) $\langle \vec{u} | \vec{u} \rangle \geq 0$, equality only for $\vec{u} = 0$.

<u>NOTE</u> (ii) and (iii) $\Rightarrow \langle a\vec{u} | \vec{v} \rangle = a^* \langle \vec{u} | \vec{v} \rangle$

Def: 1) The NORM of a vector is: $\boxed{\|\vec{v}\| \equiv \sqrt{\langle \vec{v} | \vec{v} \rangle}}$ ("length")

2) Two vectors $\vec{x}, \vec{y}$ are ORTHOGONAL if $\langle \vec{x} | \vec{y} \rangle = 0$.

Ex: 1) $\left.\begin{array}{l} \vec{x} = \sum_{i=1}^{N} x_i \vec{e}_i \\ \vec{y} = \sum_{i=1}^{N} y_i \vec{e}_i \end{array}\right\} \langle \vec{x} | \vec{y} \rangle = \sum_{i=1}^{N} x_i^* y_i$

(where $\vec{e}_i$ are basis vectors of $\mathbb{R}^n$ (or $\mathbb{C}^n$).)

2) ℓ^2-space: $\langle \vec{X} | \vec{Y} \rangle = \sum_{k=1}^{\infty} X_k^* Y_k$

3) L^2-space: $\langle f | g \rangle = \int_{-\infty}^{\infty} dx\, f(x)^* g(x) \longrightarrow$ What we use!

Theorem: 1) Schwartz Inequality: $|\langle \vec{u} | \vec{v} \rangle| \leq \|\vec{u}\| \cdot \|\vec{v}\|$

Proof: if $\vec{u}, \vec{v} = 0$, the equality holds trivially.
If not, let $\vec{z} = a\vec{u} + b\vec{v}$, and note that $\langle \vec{z} | \vec{z} \rangle \geq 0 \ \forall a, b$.

But
$$\begin{aligned} \langle \vec{z} | \vec{z} \rangle &= \langle a\vec{u} + b\vec{v} | a\vec{u} + b\vec{v} \rangle \\ &= aa^* \langle \vec{u} | \vec{u} \rangle + ab^* \langle \vec{v} | \vec{u} \rangle + a^* b \langle \vec{u} | \vec{v} \rangle + b^* b \langle \vec{v} | \vec{v} \rangle \\ &\geq 0 \end{aligned}$$

Now chose $a = \langle \vec{v} | \vec{v} \rangle$ and divide by $\langle \vec{v} | \vec{v} \rangle$:

$$\langle \vec{u} | \vec{u} \rangle \langle \vec{v} | \vec{v} \rangle + b \langle \vec{u} | \vec{v} \rangle + b^* \langle \vec{v} | \vec{u} \rangle + bb^* \geq 0$$

Now chose $b = -\langle \vec{u} | \vec{v} \rangle$ ($b^* = -\langle \vec{v} | \vec{u} \rangle$) and we have

$$\langle \vec{u} | \vec{u} \rangle \langle \vec{v} | \vec{v} \rangle \geq \langle \vec{u} | \vec{v} \rangle \langle \vec{v} | \vec{u} \rangle$$

OR

$$\|\vec{u}\|^2 \|\vec{v}\|^2 \geq |\langle \vec{u} | \vec{v} \rangle|^2 \ . \quad \text{QED.}$$

Theorem: Triangle Inequality: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Proof: $\|\vec{x} + \vec{y}\|^2 = \langle \vec{x} + \vec{y} | \vec{x} + \vec{y} \rangle = \langle \vec{x} | \vec{x} \rangle + \langle \vec{x} | \vec{y} \rangle + \langle \vec{y} | \vec{x} \rangle + \langle \vec{y} | \vec{y} \rangle$

But Schwartz $\Rightarrow |\langle \vec{x} | \vec{y} \rangle| \leq \|\vec{x}\| \, \|\vec{y}\|$, Similar with $\langle \vec{y} | \vec{x} \rangle$

So: $\|\vec{x} + \vec{y}\|^2 \leq \|\vec{x}\| + \|\vec{y}\| + 2\|\vec{x}\| \cdot \|\vec{y}\| = (\|\vec{x}\| + \|\vec{y}\|)^2$

$\therefore \ \|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$. QED.

Def: An ORTHONORMAL BASIS is a basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ where all of the basis elements are orthogonal and normalized:

$$\langle \vec{e}_i | \vec{e}_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Ex: i) "Standard Basis of $\mathbb{R}^2$": $\{(1,0), (0,1)\}$

$\{(1,0), (1,1)\}$ is a basis for $\mathbb{R}^2$, but is not ON.

$\Rightarrow$ In an ON basis:
$$\langle \vec{u} | \vec{v} \rangle = \left\langle \sum_i u_i \vec{e}_i \middle| \sum_j v_j \vec{e}_j \right\rangle = \sum_{i,j} u_i^* v_j \underbrace{\langle \vec{e}_i | \vec{e}_j \rangle}_{\delta_{ij}} = \sum_i u_i^* v_j$$

Also: $\langle \vec{e}_i | \vec{u} \rangle = \langle \vec{e}_i | \sum_j u_j \vec{e}_j \rangle = \sum_j u_j \langle \vec{e}_i | \vec{e}_j \rangle = u_i$

Then in this basis, we can write the vector $\vec{u}$ as a COLUMN VECTOR:

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \langle \vec{e}_1 | \vec{u} \rangle \\ \langle \vec{e}_2 | \vec{u} \rangle \\ \vdots \\ \langle \vec{e}_n | \vec{u} \rangle \end{pmatrix}$$

From now on, I will write $\vec{u}$ as a column vector in this way. Notice that the actual value of u_i depends on the choice of basis.

Def: A <u>LINEAR TRANSFORMATION</u> is a mapping between vector spaces

$$T: V_1 \rightarrow V_2 \ni T(a\vec{x}+b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$$

~~FACT: A linear tran If V_1 has dim =~~

FACT: If $\left.\begin{array}{l} \dim V_1 = n_1 \\ \dim V_2 = n_2 \end{array}\right\}$ then $T: V_1 \rightarrow V_2$ can be represented by an $(n_1 \times n_2)$ matrix, denoted $m(T)$.

Now $T(\vec{x}) \longleftrightarrow [m(T)]\vec{x}$, where $[m(T)]$ is a matrix and $\vec{x}$ is a column vector.

Recall: Matrix Multiplication $(AB)_{ik} = \sum_j A_{ij} B_{jk}$

In the above case: $(m(T)\vec{x})_i = \sum_j [m(T)]_{ij} \vec{x}_j$

Def: Consider 2 (distinct) bases for V ($E = \{\vec{e}_1, \ldots, \vec{e}_n\}$, $F = \{\vec{f}_1, \ldots, \vec{f}_n\}$)

and let T be a linear transformation $T: V \rightarrow V$. Now let S represent the transformation that takes $E \rightarrow F$: $\sum S_{ij} \vec{e}_j = \vec{f}_i$ and let $A \equiv m(T)$ in basis E and $B \equiv m(T)$ in basis F.

Then: $\boxed{B = S^{-1}AS}$ A and B are called <u>SIMILAR MATRICES</u>.

$$\begin{tikzcd}
\text{Basis } E: & V \arrow[r, "T", "(A)"'] & V \arrow[d, "S^{-1}"] \\
\text{Basis } F: & V \arrow[u, "S"] \arrow[r, "T", "(B)"'] & V
\end{tikzcd}$$

is commutative.

Def: Let A be an $n \times n$ matrix. A scalar λ is called an EIGENVALUE of A if $\exists$ a nonzero vector $\vec{v} \ni$

$$A\vec{v} = \lambda \vec{v} \quad \text{or equivalently} \implies (A - \lambda \mathbb{1})\vec{v} = 0$$

This has a nontrivial solution iff $\det(A - \lambda \mathbb{1}) = 0$ [ie: $(A - \lambda\mathbb{1})$ not invertible]

If A, B are similar, they have the SAME eigenvalues!

Th: If $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of an $n \times n$ matrix with corresponding EIGENVECTORS $\vec{v}_1, \ldots, \vec{v}_n$, then $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are linearly indep't.

Def: An $n \times n$ matrix A is DIAGONALIZABLE iff A has n linearly indep't eigenvectors. Let $\vec{v}_i$ be these eigenvectors, and define

$D \equiv$ matrix with each column $= \vec{v}_i$:
$$D = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

Using $A\vec{v}_i = \lambda_i \vec{v}_i$:
$$AD = (A\vec{v}_1, A\vec{v}_2, \cdots, A\vec{v}_n) = (\lambda_1\vec{v}_1, \lambda_2\vec{v}_2, \ldots, \lambda_n\vec{v}_n)$$
$$= (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = D\Lambda$$

where $\Lambda \equiv \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \implies \boxed{\Lambda = D^{-1} A D}$ SIMILAR!

Def: A matrix M ~~[illegible]~~ has an ADJOINT $M^{\dagger}$ which is the transpose conjugate of M: $\boxed{(M^{\dagger})_{ij} = M^{*}_{ji}.}$

A matrix is SELF-ADJOINT if $M^{\dagger} = M$.

↳ Some sloppy people say "Hermitian" but that is actually a stronger constraint. However, in this course, we will be sloppy!

↳ We're not lying to you, though!

⇒ Adjoint operators work to the LEFT: $(A\vec{v})^{\dagger}_i = (\vec{v}^{*} A^{\dagger})_i = \sum_j v_j^{*} A^{\dagger}_{ji}$

I will now change notation to Quantum Mechanics:

$\vec{v} \longmapsto |v\rangle$ (column matrix) "ket" $\vec{v}^{\dagger} \longmapsto \langle v|$ (Row matrix) "bra"

Then the inner product is: $\langle v|v\rangle = v^{\dagger} v$ as expected.

Now: $A|v\rangle = \lambda |v\rangle$ and $\langle v| A^{\dagger} = \lambda^{*} \langle v|$

Theorem: a) Any eigenvalue of a Hermitian matrix is real.
b) Eigenvectors with distinct eigenvalues are orthogonal.

Proof: $A|v_i\rangle = \lambda_i |v_i\rangle \Rightarrow$

$$\begin{aligned} &\langle v_j|A|v_i\rangle = \lambda_i \langle v_j|v_i\rangle \\ - \; &\langle v_j|A^{\dagger}|v_i\rangle = \lambda_j^{*} \langle v_j|v_i\rangle \\ \hline \text{Since } A = A^{\dagger} \longrightarrow \; & 0 = (\lambda_i - \lambda_j^{*}) \langle v_j|v_i\rangle \end{aligned}$$

a) $(i=j)$: $(\lambda_i - \lambda_i^{*}) \|v\| = 0$, and $v \neq 0 \Rightarrow \lambda_i = \lambda_i^{*} \Rightarrow \lambda_i$ Real ✓

b) $(i \neq j)$: $(\lambda_i - \lambda_j^{*}) \langle v_j|v_i\rangle = 0$, $\lambda_i \neq \lambda_j^{*} \Rightarrow \langle v_j|v_i\rangle = 0$ ✓

Def: An $n \times n$ matrix U is UNITARY if its column vectors form an orthonormal set in $\mathbb{C}^n$, ie: $\boxed{U^{\dagger} U = \mathbb{1}} \Leftrightarrow \underline{U^{\dagger} = U^{-1}}$

Th: If A is Hermitian, $\exists\, U$ unitary that diagonalizes A:

$$\Lambda_A = U^{\dagger} A U.$$