

# Quantum Mechanics

## 171.303

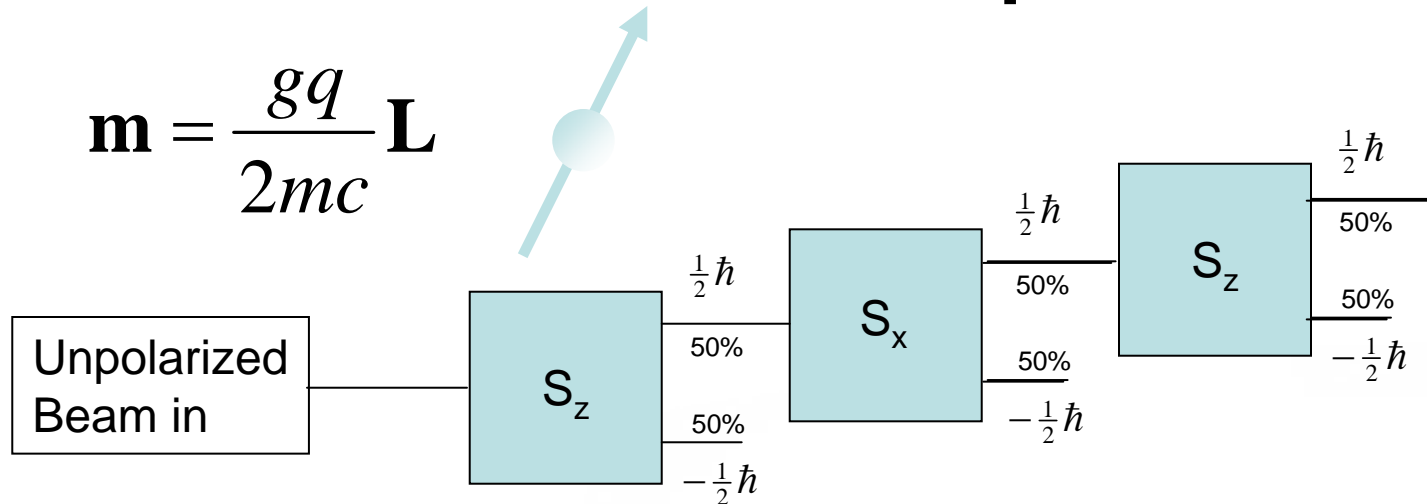
Summary of Fall Semester

By Collin Broholm



# The Stern Gerlach Experiment

$$\mathbf{m} = \frac{gq}{2mc} \mathbf{L}$$



$$|\psi\rangle = c_+ |+\ z\rangle + c_- |-\ z\rangle$$

Resolve quantum state in components

$$1 = \langle \psi | \psi \rangle = |c_+|^2 + |c_-|^2$$

Normalization for probabilistic interpretation

$$|c_+|^2 = \text{Probability that measurement of } S_z \text{ yields } \frac{1}{2} \hbar$$

$$1 = |+\ z\rangle\langle + z| + |-\ z\rangle\langle - z|$$

Completeness

$$\langle + z | - z \rangle = \langle - z | + z \rangle = 0$$

Orthogonality

Generalize to  
multidimensional  
"Hilbert space"

# Representations and Operators

$$\left. \begin{aligned} |\pm x\rangle &= \frac{1}{\sqrt{2}} (|+z\rangle \pm |-z\rangle) \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \\ |\pm y\rangle &= \frac{1}{\sqrt{2}} (|+z\rangle \pm i|-z\rangle) \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \end{aligned} \right\} \text{“Representation” of Ket in chosen basis}$$

$$\hat{A}|\psi\rangle = |\phi\rangle \rightarrow \sum_{ij} \langle i|\hat{A}|j\rangle \langle j|\psi\rangle = \sum_i \langle i|\phi\rangle$$

$$\langle \psi|\hat{A}^+ = \langle \phi| \rightarrow \sum_{ij} \langle \phi|j\rangle \langle j|\hat{A}|i\rangle^* = \sum_i \langle \phi|i\rangle$$

$$\hat{P}_{\pm} = |\pm z\rangle\langle \pm z| \quad \text{Projects component: } \hat{P}_+(c_+|+z\rangle + c_-|-z\rangle) = c_+|+z\rangle$$

$$\hat{P}_+ \rightarrow \begin{pmatrix} \langle +z|\hat{P}_+|+z\rangle & \langle +z|\hat{P}_+|-z\rangle \\ \langle -z|\hat{P}_+|+z\rangle & \langle -z|\hat{P}_+|-z\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

# Change of basis

For ket (shown here for two state basis)

$$\begin{pmatrix} \langle +u | \psi \rangle \\ \langle -u | \psi \rangle \end{pmatrix} = \underbrace{\begin{pmatrix} \langle +u | +z \rangle & \langle +u | -z \rangle \\ \langle -u | +z \rangle & \langle -u | -z \rangle \end{pmatrix}}_{\text{Columns are coordinates of old basis states in new basis}} \begin{pmatrix} \langle +z | \psi \rangle \\ \langle -z | \psi \rangle \end{pmatrix} \equiv \mathbf{S}^+ \begin{pmatrix} \langle +z | \psi \rangle \\ \langle -z | \psi \rangle \end{pmatrix}$$

Columns are coordinates of  
old basis states in new basis

For Operator

$$\mathbf{A}(\text{new basis}) = \mathbf{S}^+ \mathbf{A}(\text{old basis}) \mathbf{S}$$

New and old states are related by **unitary** transformation

$$|\pm u\rangle = \hat{R} |\pm z\rangle \quad (\text{where } \hat{R}^+ \hat{R} = 1 \text{ to preserve normalized states})$$

Then  $\mathbf{S}$  is representation of  $\hat{R}$  in new and old basis

# Rotation, Angular Momentum & observables

$$\hat{R}(\phi \hat{\mathbf{n}}) = \exp(-i\hat{J}_n \phi / \hbar)$$

$\hat{J}_n$  is Hermitian ( $\hat{J}_n^+ = \hat{J}_n$ ) with dimension of angular momentum

We determine the action of  $\hat{R}(\phi \hat{\mathbf{n}})$  through Taylor expansion & find

$$\hat{J}_n |\pm n\rangle = \pm \frac{1}{2} \hbar |\pm n\rangle \quad \longrightarrow \quad \begin{array}{l} \text{Eigen states } |\pm n\rangle \\ \text{Eigen values } \pm \frac{1}{2} \hbar \end{array}$$

Infer that  $\hat{J}_n$  is angular momentum projection operator

$$\hat{S}_x \rightarrow \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y \rightarrow \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{S}_z \rightarrow \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\langle J_x \rangle = \langle \psi | \hat{J}_x | \psi \rangle = (\langle \psi | + z \rangle, \langle \psi | - z \rangle) \frac{1}{2} \hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \langle + z | \psi \rangle \\ \langle - z | \psi \rangle \end{pmatrix}$$

# Angular Momentum Algebra

Order of rotations matter which in operator language implies:

$$[\hat{R}(\delta\phi \hat{x}), \hat{R}(\delta\phi \hat{y})] = \hat{R}(\delta\phi \hat{z}) - 1$$

This implies that angular momentum operators don't commute:

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z \quad \Delta J_x \Delta J_y \geq \left| \frac{1}{2i} \langle [\hat{J}_x, \hat{J}_y] \rangle \right| = \frac{\hbar}{2} |\langle J_z \rangle|$$

We define raising and lowering operators:

$$\hat{J}_{\pm} = \hat{J}_x \pm i\hat{J}_y \quad [\hat{J}_z, \hat{J}_{\pm}] = \pm\hbar\hat{J}_{\pm}$$

From which we derive angular momentum eigenstate spectrum

$$\hat{J}^2 |j, m\rangle = j(j+1)\hbar^2 |j, m\rangle \quad \hat{J}_{\pm} |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle$$

$$\hat{J}_z |j, m\rangle = m\hbar |j, m\rangle$$

Representation of  $\hat{J}_x$  and  $\hat{J}_y$  for any j are readily  
Derived through the representation of the  $\hat{J}_{\pm}$  using

$$\hat{J}_x = \frac{1}{2} (\hat{J}_+ + \hat{J}_-) \quad \hat{J}_y = \frac{1}{2i} (\hat{J}_+ - \hat{J}_-)$$

# The Schrodinger Equation

Time evolution described by unitary transformation of t=0 wave function:

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$$

$$\hat{U}(t) = \exp(-i\hat{\mathcal{H}}t/\hbar)$$

From this derive Schrodinger differential equation for

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{\mathcal{H}}|\psi(t)\rangle$$

And Ehrenfest equation for evolution of averages of observables

$$\frac{d}{dt}\langle \hat{A} \rangle = \frac{i}{\hbar} \langle \psi(t) | [\hat{\mathcal{H}}, \hat{A}] | \psi(t) \rangle$$

Time required to appreciate change depends on energy spread:

$$\Delta E \Delta t \geq \frac{1}{2} \hbar$$

For spin in a field along z-axis  $\hat{\mathcal{H}} = -\frac{gq}{2mc} B \hat{S}_z$  and  $\hat{U}(t)$  is rot. op.

$$\langle S_x \rangle = \frac{\hbar}{2} \cos \omega t \quad \langle S_y \rangle = \frac{\hbar}{2} \sin \omega t \quad \langle S_z \rangle = 0$$

# Combined Angular Momentum etc.

Two spins can be described as product of eigenstates for original spins or as sum of combined angular momentum states:

$$|JM\rangle = \sum_{m_1+m_2=M} C_{m_1 m_2 M}^{j_1 j_2 J} |j_1 m_1\rangle |j_2 m_2\rangle$$

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{J, M=m_1+m_2} \bar{C}_{M m_1 m_2}^{J j_1 j_2} |JM\rangle$$

Singlet state is rotationally invariant:  $\hat{R}(\phi \hat{\mathbf{n}})|0,0\rangle = |0,0\rangle$

Disintegration of singlet reveals fundamental nature of QM uncertainty

$$\boxed{\hat{S}_a} \longleftarrow \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \longrightarrow \boxed{\hat{S}_b}$$

Hidden variable theory obeys Bell's inequality. QM does NOT hence

- QM contains complete information of physical state
- QM is a non-local theory

# Wave mechanics in one dimension

Translation operator  $\hat{T}(x) = \exp(-i\hat{p}_x x / \hbar)$

$\hat{p}_x$  is momentum operator for x-direction  $\hat{p}_x |p_x\rangle = p_x |p_x\rangle$

$$[\hat{x}, \hat{p}_x] = i\hbar \Rightarrow \Delta x \Delta p_x \geq \frac{\hbar}{2}$$

ENTITY	POSITION SPACE	MOMENTUM SPACE
$ \psi\rangle$	$\psi(x) = \langle x   \psi \rangle$	$\Phi(p) = \langle p   \psi \rangle$
$\hat{p} \psi\rangle$	$\frac{\hbar}{i} \frac{\partial}{\partial x} \psi(x)$	$p\Phi(p)$
$\psi(x)$	$\psi(x)$	$\frac{1}{\sqrt{2\pi\hbar}} \int \exp(ipx/\hbar) \Phi(p) dp$
$\Phi(p)$	$\frac{1}{\sqrt{2\pi\hbar}} \int \exp(-ipx/\hbar) \psi(x) dx$	$\Phi(p)$

$|\psi(x)|^2$  Is probability of measuring particle between x and x+dx

$|\Phi(p)|^2$  Is probability of measuring momentum between p and p+dp

Time propagation of free particle wave function  $\Phi(p, t) = \exp\left(-i \frac{p^2}{2m} \frac{t}{\hbar}\right) \Phi(p, 0)$

# Scattering and bound states in 1D

Bound states have energy quantization and vanish at infinity

Scattering states have continuous energy distribution and travel to infinity

Bound states occur for  $E < V(\pm \infty)$

Scattering states occur for  $E > V(\pm \infty)$

If  $V(x) > V(\pm \infty)$  there are only scattering states

If  $V(x) < V(\pm \infty)$  there can be bound and scattering states

Solve time independent Schrodinger equation:

$$\hat{H}\psi(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_E(x) + V(x)\psi_E(x) = E\psi_E(x)$$

Solutions must satisfy boundary conditions that often give quantization:

$$\psi(0^-) = \psi(0^+)$$

Wave function continuous

$$-\frac{\hbar^2}{2m} (\psi'(0^+) - \psi'(0^-)) + \psi(0) \int_{0^-}^{0^+} V(x) dx = 0$$

Wave function slope discontinuity reflects any divergences in  $V(x)$

Putting in time dependence:  $\psi_E(x, t) = \exp(-iEt / \hbar) \psi_E(x, t = 0)$

# The 1D Harmonic Oscillator

$$\hat{\mathcal{H}}|\psi\rangle = \left( \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2 \right) |\psi\rangle = E|\psi\rangle$$

Can diagonalize by introducing

$$\begin{aligned} \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} + \frac{i}{m\omega} \hat{p} \right) \\ \hat{a}^+ &= \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} - \frac{i}{m\omega} \hat{p} \right) \end{aligned} \quad \longrightarrow \quad \hat{\mathcal{H}} = \hbar\omega \left( \hat{a}^+ \hat{a} + \frac{1}{2} \right)$$

Commutation relations determine the algebra eigenstates and values:

$$\begin{aligned} [\hat{a}, \hat{a}^+] &= 1 & E_n &= \hbar\omega \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle & \psi_n(x) &= \frac{1}{\sqrt{n!}} \langle x | (\hat{a}^+)^n | 0 \rangle = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(y) e^{-y^2/2} \\ \hat{a}^+|n\rangle &= \sqrt{n+1}|n+1\rangle & y &= \sqrt{\frac{m\omega}{\hbar}} x \end{aligned}$$

# WKB Approximation

An approximation to  $\psi(x)$  when  $V(x)$  is slowly varying

Define  $p^2(x) = 2m(E - V(x))$

For classical regime  $p^2(x) > 0$ :

$$\psi(x) \propto \frac{1}{\sqrt{p(x)}} \exp(\pm i \frac{1}{\hbar} \int p(x) dx)$$

For non classical regime  $p^2(x) < 0$ :

$$\psi(x) \propto \frac{1}{\sqrt{|p(x)|}} \exp(\pm \frac{1}{\hbar} \int |p(x)| dx)$$

Quantization conditions for WKB bound states ( $n=1,2,3,\dots$ ):

$$\int_{x_1}^{x_2} p(x) dx = \begin{cases} n\pi\hbar & \text{Two infinite boundaries} \\ (n - \frac{1}{4})\pi\hbar & \text{One infinite boundary} \\ (n - \frac{1}{2})\pi\hbar & \text{No infinite boundaries} \end{cases}$$

# Good Luck with the Exam!

- Practice exams will be provided
- Problem solving session with Andrew TBA
- Office hours with Collin this afternoon and by appt.
- Final exam on 12/20 9-noon in room 278
- See you next semester!

