Gravitational time dilation.

Recall the proper time:

$$d\tau = \sqrt{g_{00}}\, dx^0$$

~~and the~~ and take a periodic phenomenon of frequency ν. (e.g. an atomic transition).

Static gravitational field, two points, x_1, x_2 at rest w.r.t. each other.

$$dt_i = d\tau \left(g_{00}(x_i) \right)^{-1/2}$$

Periods:

$$T_i = \int dt_i = \int d\tau \left(g_{00}(x_i) \right)^{-1/2}$$

$$= T_0 \left(g_{00}(x_i) \right)^{-1/2}$$

Hence

$$\nu_i = \nu_0 \left(g_{00}(x_i) \right)^{1/2}$$

Cannot compare to "standard clock": gravity affects all "clocks" equally.

Measurable:

$$\frac{\nu_1}{\nu_2} = \frac{g_{00}(x_1)}{g_{00}(x_2)}$$

The Schwarzschild metric: black holes.

[Newtonian analogy: Find a solution of $\nabla^2\phi = 0$ with a singularity @ the origin, s.t. the solution is spherically symmetric.]

Try:

$$ds^2 = A(r)\,dt^2 - B(r)\,dr^2 - r^2\,d\Omega^2 \qquad \left(d\Omega^2 = d\theta^2 + \sin^2\theta\,d\varphi^2\right)$$

Insert into $\int d^4x \sqrt{-g}\,R$ and extremize.

$$\Rightarrow\ ds^2 = dt^2\left(1-\frac{R_s}{r}\right) - r^2 d\Omega^2 - \frac{dr^2}{1-\frac{R_s}{r}}$$

Here $R_s = 2MG$, the Schwarzschild radius.

NB: if $r \leq R_s$, dt^2 and dr^2 change roles: event horizon.

The question is: where is R_s? (In- or outside the star?)

$$\frac{\nu(r)}{\nu(\infty)} = \left(1-\frac{R_s}{r}\right)^{1/2}$$

∞ red shift @ R_s

Numerical data. $(\hbar = c = 1)$

$G = \frac{1}{M_P^2}$ $M_P \approx 10^{19}$ GeV $(1.2 \times 10^{19}$ GeV$)$

[In old-fashioned units,
$M_P^2 = \frac{\hbar c}{G}$]

$$\frac{\hbar}{m_N c} \approx 2 \times 10^{-14}\text{cm} \longrightarrow \frac{1}{m_N} \quad (\hbar = c = 1)$$

$$\frac{1}{M_P} \approx 10^{-33}\text{cm}$$

~~Schwartz~~
Schwarzschild radius

$$R_S = \frac{2M}{M_P}\frac{1}{M_P}$$

$$\approx \frac{2M}{10^{-5}\text{kg}} \times 10^{-33}\text{cm}$$

R_S for Earth: $R_{S\oplus} \approx 4$ mm

How do I obtain Schwarzschild?

Put $A(r) = e^{N(r)}$, $B(r) = e^{L(r)}$

$(x^0 = t,\ x^1 = r,\ x^2 = \theta,\ x^3 = \phi)$

$$g_{\mu\nu} = \begin{pmatrix} e^N & & & \\ & -e^L & & \\ & & -r^2 & \\ & & & -r^2 \sin^2\theta \end{pmatrix}$$

Nonvanishing connection coefficients:

$$\Gamma^0_{01} = \frac{1}{2} N' = \Gamma^0_{10} \quad \left(N' = \frac{dN}{dr}\right)$$

$$\Gamma^1_{00} = \frac{1}{2} N' e^{N-L}$$

$$\Gamma^1_{11} = \frac{1}{2} L'$$

$$\Gamma^1_{22} = -r e^{-L}$$

$$\Gamma^1_{33} = -r \sin^2\theta\, e^{-L}$$

$$\Gamma^2_{12} = \frac{1}{r} = \Gamma^2_{21}$$

$$\Gamma^2_{33} = -\sin\theta \cos\theta$$

$$\Gamma^3_{13} = \frac{1}{r} = \Gamma^3_{31}$$

$$\Gamma^3_{23} = \cot\theta = \Gamma^3_{32}$$

Einstein : $R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R = 0$ (Vacuum eq.)

(0) $R_0^0 - \frac{1}{2}R = -e^{-L}\left(\frac{L'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} = 0$

(1) $R_1^1 - \frac{1}{2}R = e^{-L}\left(\frac{N'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0$

(2) $R_2^2 - \frac{1}{2}R = e^{-L}\left(\frac{N''}{2} - \frac{L'N'}{4} + \frac{N'^2}{4} + \frac{N'-L'}{2r}\right) = 0$

(3) $R_3^3 - \frac{1}{2}R = e^{-L}\left(\frac{N''}{2} - \frac{L'N'}{4} + \frac{N'^2}{4} + \frac{N'-L'}{2r}\right) = 0$

(R_2^2 and R_3^3 are the same)

Rewrite (0) : $e^{-L}(-rL'+1) = 1$

Solution $e^L = \frac{1}{1-\frac{c}{r}}$

Subtract (0) from (1) : $L' = -N'$

$\Rightarrow L = -N + \text{const}$

The metric is asymptotically flat:

$L \to 0$ ~~[illegible]~~, $M \to 0$ $(r \to \infty)$

hence const = 0

$$\Rightarrow \quad e^N = 1 - \frac{c}{r}$$

Hence:

$$ds^2 = \left(1-\frac{c}{r}\right)dt^2 - \frac{dr^2}{1-\frac{c}{r}} - r^2 d\Omega^2$$

For $r \to \infty$, $g_{00} \approx 1 + \frac{2\phi}{c^2}$

($= 1 + 2\phi$ in natural units)

$\phi = \frac{GM}{r}$, hence, $c = 2GM$

(The apparent difference between this and the quoted Newtonian limit — Notes, p. 15 — can be removed by redefining r.)

Relevance of Schwarzschild?

There are no point sources in real life!

However: collapsed star, sol'n valid outside R_s

Deflection of light

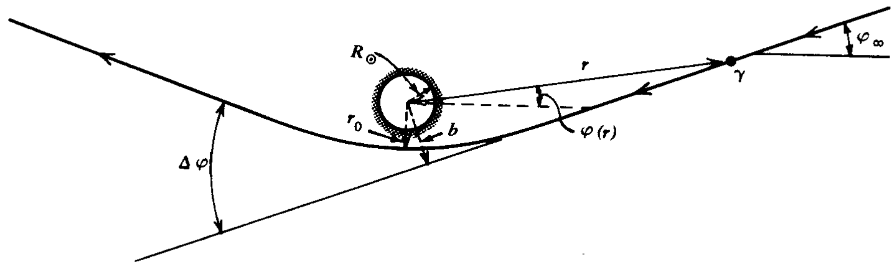


Figure 8.1 Quantities referred to in the calculation of the deflection of light by the sun. (Deflection greatly exaggerated.)

Ray optical approx.

Figure for computing the deflection of light.

At infinity: $A = B = 1$

$$b \approx r \sin(\varphi - \varphi_0) \approx r(\varphi - \varphi_0)$$

~~Velocity~~ $V \approx \dfrac{dr}{dt}$

~~Conserved quantities~~

(planar motion)

Eq. of geodesic:

$$\frac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} = 0$$

(p is an arbitrary affine parameter)

Write out for

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\Omega_2^2$$

$$0 = \frac{d^2r}{dp^2} + \frac{B'}{2B}\left(\frac{dr}{dp}\right)^2 - \frac{r}{B}\left(\frac{d\theta}{dp}\right)^2 - r\frac{\sin^2\theta}{B}\left(\frac{d\varphi}{dp}\right)^2 + \frac{A'}{B}\left(\frac{dt}{dp}\right)^2 \quad (1)$$

$$0 = \frac{d^2\theta}{dp^2} + \frac{2}{r}\frac{d\theta}{dp}\frac{dr}{dp} - \sin\theta\cos\theta\left(\frac{d\varphi}{dp}\right)^2 \quad (2)$$

$$0 = \frac{d^2\varphi}{dp^2} + \frac{2}{r}\frac{d\varphi}{dp}\frac{dr}{dp} + 2\cot\theta\frac{d\varphi}{dp}\frac{d\theta}{dp} \quad (3)$$

$$0 = \frac{d^2t}{dp^2} + \frac{A'}{A}\frac{dt}{dp}\frac{dr}{dp} \quad (0)$$

Notice that $\theta = \text{const} = \frac{\pi}{2}$ is a solution of (2): planar motion.

Choose this solution.

~~Divide (3) by~~ Multiply (3) by $\frac{dp}{d\varphi}$ and (0) by $\frac{dp}{dt}$. This gives two conservation laws (NB $\cot\frac{\pi}{2} = 0$) ~~etc~~ :

$$\frac{d}{dp}\left(\ln\frac{d\varphi}{dp} + \ln r^2\right) = 0 \qquad (a)$$

$$\frac{d}{dp}\left(\ln\frac{dt}{dp} + \ln A\right) = 0 \qquad (b)$$

(a) gives $\frac{d\varphi}{dp} = (\text{const})\frac{1}{r^2}$

~~and (const) can be absorbed into the def. of p.~~

(b) gives (by a choice of p)

$$\frac{dt}{dp} = \frac{1}{A}$$

From (a): $r^2\frac{d\varphi}{dp} = J$

("angular momentum")

There remains one eq:

$$0 = \frac{d^2r}{dp^2} + \frac{B'}{B}\left(\frac{dr}{dp}\right)^2 - \frac{J^2}{r^3 B} + \frac{A'}{3BA^2} \qquad (c)$$

Multiply this by $2B\frac{dr}{dp}$:

$$\frac{d}{dp}\left\{ B\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{A} \right\} = 0$$

$$\Rightarrow \quad B\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{A} = E = \text{const}$$

Use this and the form of ds^2 to get

$$ds^2 = E\, dp^2$$

so $E = 0$ for photons and $E > 0$ for ~~a~~ massive particles.

Eliminate p in favor of t by using

$$\frac{dt}{dp} = \frac{1}{A}$$

$$r^2 \frac{d\varphi}{dt} = JA$$

$$\frac{B}{A^2}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{r^2} - \frac{1}{A} = E$$

Deflection of light: $E=0$, unbound orbit.

$$A = 1 - \frac{2GM}{r}, \quad B = \frac{1}{1-\frac{2GM}{r}} \approx 1 + \frac{2GM}{r}$$

Note that b is the impact parameter in the absence of gravity. If r_0 is the distance of closest approach with gravity present, for distant orbits:

$$r \gg R_\odot \gg R_S,$$

$$r_0 \approx b$$

For light: $E=0$

$$r^2 \frac{d\varphi}{dt} = JA = J\left(1 - \frac{2GM}{r}\right)$$

Use the Figure and choose $\varphi_\infty = 0$

$$b \approx r\varphi$$

Equation for the orbit: eliminate t (or p):

$$\frac{B}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2} - \frac{1}{J^2 A} = \left(\frac{E}{J^2}\right) = 0$$

(because $E = 0$)

$$d\varphi^2\left(\frac{1}{r^2} - \frac{1}{AJ^2}\right) + \frac{B}{r^4}\,dr^2 = 0$$

$$d\varphi = \frac{dr\; B^{1/2}}{r^2\left(\frac{1}{AJ^2} - \frac{1}{r^2}\right)^{1/2}}$$

Determine J.

At r_0 $\frac{dr}{d\varphi} = 0$, thus

$$\frac{1}{J^2 A} = \frac{1}{r_0^2} \approx \frac{1}{b^2}$$

$$d\varphi \approx \frac{dr\; B^{1/2}}{r^2\left(\frac{1}{b^2} - \frac{1}{r^2}\right)^{1/2}}$$

This can be integrated in terms of elementary functions, using the expansion:

$$B^{1/2} = \frac{1}{\sqrt{1-\frac{2GM}{r}}} \approx 1 + \frac{GM}{r} + \cdots$$

Then

$$\varphi \approx \int_b^\infty \frac{dr}{r^2\left(\frac{1}{b^2}-\frac{1}{r^2}\right)^{1/2}}\left(1+\frac{GM}{r}\right)$$

Clearly, the first term is the change without gravity and the term $\propto GM$ gives the gravitational ~~defl~~ deflection

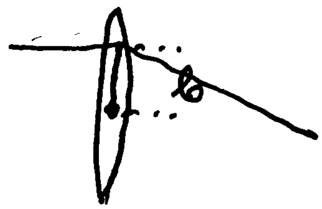
$$\Delta\varphi = 2GM\int_b^\infty \frac{dr}{r^3\left(\frac{1}{b^2}-\frac{1}{r^2}\right)^{1/2}}$$

$$= \frac{4GM}{b}$$

(One estimates: $b \approx R_\odot$ for grazing incidence. Numerically $\Delta\varphi = 1.75''$.)

Gravitational "lensing".

Not a real lens. For an optical lens, $\Delta\varphi \propto b$:

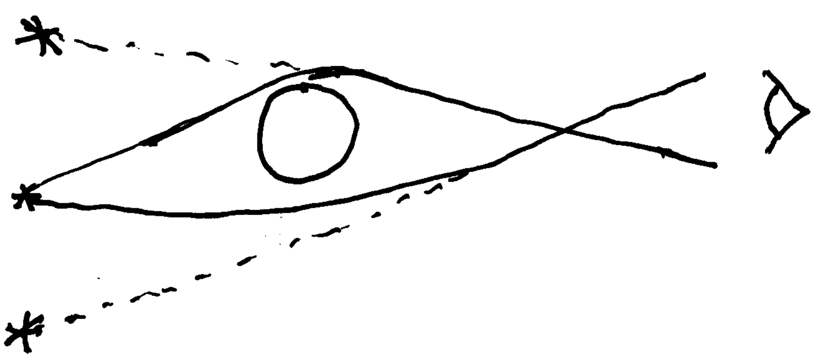


However for gravitational deflection:

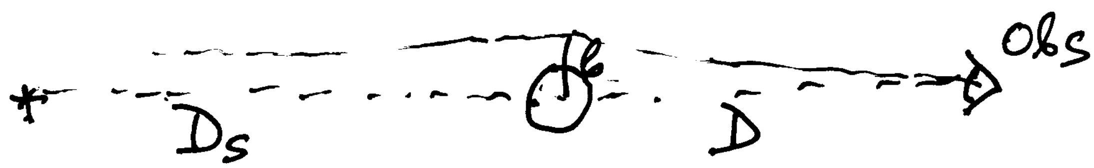
$$\Delta\varphi = \frac{4GM}{b}$$

(No ~~real~~ image formation because there is no well defined focal length.)

Multiple images:



Einstein ring: cylindrically symmetric arrangement



$D_s \gg D$

The incident rays are nearly parallel to each other.

The angle needed to reach the observer is:

$$\varphi = b/D = \frac{4GM}{b}$$

$$b^2 = 4GMD$$

$$b = \sqrt{4GMD}$$

$$\varphi = \frac{b}{D} = \sqrt{\frac{4GM}{D}}$$

In the Milky Way, a typical star has $M \approx M_\odot$, $d \approx 10^4$ ly

(1 ly $\simeq 10^{16}$ m)

$$\frac{b}{D} \simeq 8 \times 10^{-9} \text{ radian} \simeq 2 \times 10^{-3} \text{ arcsec}$$

(Too small to be observable)

For a galaxy $M \approx 10^{11} M_\odot$, $D \approx 10^{10}$ ly

$$\frac{b}{D} \approx 1 \text{ arcsec}$$

(Barely possible for an optical telescope easy for a radio telescope.)

For a galaxy, multiple images are possible: the galaxy is essentially transparent;

$R_* \ll d$ (distance between stars)

The shape of extended objects is distorted

<u>Microlensing</u>: stars move in galaxies; hence, time dependent fluctuations.
Dark matter distorts the image:
a disk is distorted into an arc.
"Arclets" (many) seen in some lenses.

Important from obs. point of view:
lensing is achromatic.