Advanced Statistical Mechanics 171.703

Homework Assignment 1

Due date Friday, February 6

Reading. Chapters 1–9 in Landau.

Problem 1. Invariant measures in phase space.

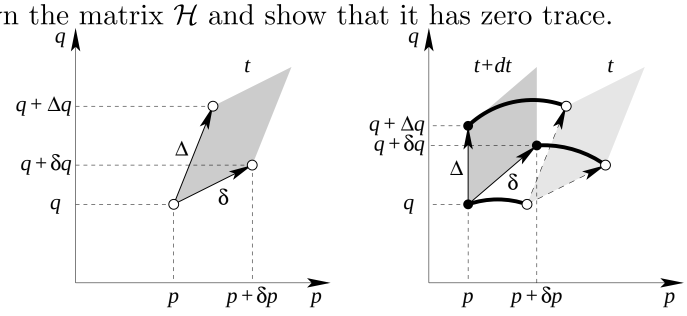
(a) Take an arbitrary point $(\mathbf{p}, \mathbf{q}) \equiv (p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n)$ in phase space and a point in its vicinity $(\mathbf{p}', \mathbf{q}') = (\mathbf{p} + \delta\mathbf{p}, \mathbf{q} + \delta\mathbf{q})$. As the system evolves in time, the two points remain close to each other. By using equations of the Hamiltonian dynamics derive the equations of motion for the infinitesimal vector $\delta(t) \equiv (\delta\mathbf{p}(t), \delta\mathbf{q}(t))$:

$$\begin{aligned} \delta\dot{p}_i &= -\sum_{j=1}^{n} \left(\frac{\partial^2 H}{\partial p_j \partial q_i}\, \delta p_j + \frac{\partial^2 H}{\partial q_j \partial q_i}\, \delta q_j \right), \\ \delta\dot{q}_i &= +\sum_{j=1}^{n} \left(\frac{\partial^2 H}{\partial p_j \partial p_i}\, \delta p_j + \frac{\partial^2 H}{\partial q_j \partial p_i}\, \delta q_j \right). \end{aligned}$$

(b) Show that evolution of the vector $\delta(t)$ over an infinitesinal time interval $\mathrm{d}t$ is given by a linear transformation

$$\delta(t + \mathrm{d}t) = (1 + \mathcal{H}\,\mathrm{d}t)\,\delta(t). \tag{1}$$

Write down the matrix $\mathcal{H}$ and show that it has zero trace.



(c) Take another point nearby: $(\mathbf{p}'', \mathbf{q}'') = (\mathbf{p} + \Delta\mathbf{p},\ \mathbf{q} + \Delta\mathbf{q})$. The two infinitesimal displacement vectors $\delta \equiv (\delta\mathbf{p}, \delta\mathbf{q})$ and $\Delta = (\Delta\mathbf{p}, \Delta\mathbf{q})$ form a parallelogram of area

$$\Omega(\delta, \Delta) = \sum_{i=1}^{n} (\delta p_i \, \Delta q_i - \delta q_i \, \Delta p_i) = -\Omega(\Delta, \delta). \tag{2}$$

Show that the area (the 2-form) $\Omega(\delta, \Delta)$ does not vary with time.

(d) In class we have shown that, for a system with $n = 1$ degree of freedom, the area can be expressed as a determinant

$$\Omega(\delta, \Delta) = \det \begin{pmatrix} \delta p & \Delta p \\ \delta q & \Delta q \end{pmatrix},$$

that is to say, the 2-dimensional volume. For $n > 1$, higher-order invariants ($2n$-forms) can be defined. For n degrees of freedom, consider an infinitesimal parallelepiped in its $2n$-dimensional phase space formed by $2n$ infinitesimal vectors δ^i. Show that its volume

$$\Omega_{2n}(\delta^1, \delta^2, \ldots, \delta^{2n}) = \det \begin{pmatrix} \delta^1 p_1 & \delta^2 p_1 & \ldots & \delta^{2n} p_1 \\ \vdots & \vdots & & \vdots \\ \delta^1 p_n & \delta^2 p_n & \ldots & \delta^{2n} p_n \\ \delta^1 q_1 & \delta^2 q_1 & \ldots & \delta^{2n} q_1 \\ \vdots & \vdots & & \vdots \\ \delta^1 q_n & \delta^2 q_n & \ldots & \delta^{2n} q_n \end{pmatrix}$$

remains constant in time: $\Omega_{2n}(t + \mathrm{d}t) = \Omega_{2n}(t)$. *Hint:* Use Eq. (1).

Problem 2. Consider a classical system consisting of $N \gg 1$ harmonic oscillators with the Hamiltonian

$$H(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + \frac{m\omega^2 q_i^2}{2} \right).$$

(a) Compute the number of states $\Gamma(E)$ with energies less than E. *Hint:* The volume of a $2n$-dimensional sphere of unit radius is $V_{2n} = \pi^n/n!$

(b) Determine the average value of energy $\bar{E} = \langle E \rangle$ and its fluctuation $\Delta E = (\langle E^2 \rangle - \langle E \rangle^2)^{1/2}$ in a canonical ensemble with $\rho(\mathbf{p}, \mathbf{q}) = \mathrm{const}\, e^{-H/T}$. Show that the relative fluctuation $\Delta E/\bar{E}$ is small for large N.

(c) Compute the entropy of the canonical ensemble $S = \log \Delta\Gamma$, where $\Delta\Gamma$ is the number of states accessible to the ensemble,

$$\Delta\Gamma = \left. \frac{\mathrm{d}\Gamma(E)}{\mathrm{d}E} \right|_{E=\bar{E}} \Delta E.$$