Advanced Statistical Mechanics 171.703

Homework Assignment 10 (corrected)

Due date Friday, April 23

Assigned reading. Ch. 11.10–11.13 in Pathria. Also skim Ch. 13.1.

Problem 1. Magnetic monopoles in nanorings. Frank Zhu makes nanorings, in which the vector of magnetization $\mathbf{m}$ points along the circumference. A ring has two degenerate thermodynamic states, in which the magnetization points clockwise (Fig. 1a) or anticlockwise (Fig. 1b). The free energy of the ring can be obtained by the minimizing the Ginzburg-Landau free energy

$$\mathcal{F}[\mathbf{m}(x)] = \mathcal{F}(0) + \int [-\mathbf{h}\cdot\mathbf{m} + a\,m^2 + b\,m^4/2 + c\,(\mathrm{d}m/\mathrm{d}x)^2]\,\mathrm{d}x$$

with respect to the magnetization $\mathbf{m}(x)$. Here $x = R\cos\phi$ is the azimuthal coordinate; $a < 0$, $b > 0$ and $c > 0$ are material-specific constants. Assume that the correlation length $\xi = \sqrt{c/|a|}$ is small compared to the radius R.

(a) Determine the equilibrium magnetization m_0 in zero magnetic field.

(b) With the ring in the ground state of Fig. 1b, Frank ramps up a uniform magnetic field $\mathbf{h}$ until a small domain of the opposite magnetization appears at the top (Fig. 1c). Determine the coercive field h_c.

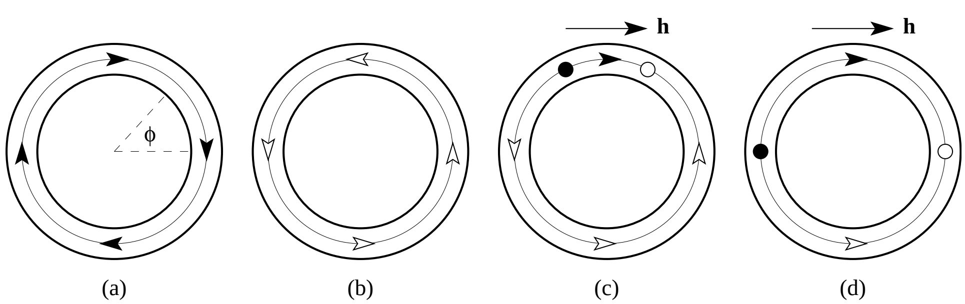

Figure 1:

(c) The two domains are separated by domain walls: a kink and an antikink (Fig. 1c). Show that a domain wall behaves like a magnetic monopole: a uniform applied field exerts a force $\mathbf{F} = q\mathbf{h}$ on it. Determine the magnetic charges q of the kink and antikink in the limit of a weak field. Express them in terms of physically transparent parameters (e.g. m_0 rather than a).

(d) Show that the new equilibrium configuration (still in an applied field) is shown in Fig. 1d.

(e) The field is reduced to a value $h < h_c$ but still points in the same direction. What happens?

(f) The field is reversed, $\mathbf{h} \mapsto -\mathbf{h}$. What happens then?

2. Robust nature of kinks. Demonstrate that the existence of solitons is not simply an artifact of the Landau potential $U(m) = am^2 + bm^4/2$. To do so, consider a general field theory in one dimension resulting from the minimization of the functional

$$\mathcal{F}[m(x)] = \int [U(m) + c(\mathrm{d}m/\mathrm{d}x)^2]\,\mathrm{d}x$$

with respect to real functions $m(x)$. Let the potential be an even function, $U(m) = U(-m)$, with exactly two minima at $m = m_0$ and $-m_0$, a maximum at $m = 0$, and no other extrema.

(a) Show that, in addition to the two uniform solutions (the vacua), there are also soliton solutions, kinks and antikinks $m_\pm(x, x_0) = \pm f(x - x_0)$, such that $f(\pm\infty) = \pm m_0$. Specify any other restrictions on $U(m)$ that you need.

(b) Show that the soliton is an excited state, i.e. that it has a positive free energy:

$$\Delta\mathcal{F} = \mathcal{F}[m_\pm(x, x_0)] - \mathcal{F}[m_0] > 0.$$

(c) Compute the free energy of a soliton for the Landau potential. Express the result in terms of physically transparent parameters.

3. Vortices in a superfluid. A Bose superfluid has a complex order parameter $\psi(\mathbf{r})$ characterizing the wavefunction of the condensate. The variational free energy of a superfluid is

$$\mathcal{F}[\psi(\mathbf{r})] = \int (a|\psi|^2 + b|\psi|^4/2 + \hbar^2|\nabla\psi|^2/2m)\,\mathrm{d}V.$$

A nonzero order parameter ψ breaks the symmetry of the free energy with respect to phase shifts, $\psi \to \psi e^{i\theta}$. The *continuous* nature of the broken symmetry in this theory leads to very different properties of solitons in it.

(a) Find *all* uniform solutions $\psi(\mathbf{r}) = \psi_0 = \text{const}$ minimizing the free energy for $a < 0$ and $b > 0$.

(b) At low temperatures, fluctuations of the *magnitude* of the order parameter can be neglected, however the fluctuations of the phase still occur. Parametrization $\psi(\mathbf{r}) = |\psi_0|e^{i\theta(\mathbf{r})}$ converts the theory of a complex field into a theory of a real field $\theta(\mathbf{r})$ defined modulo 2π. (In other words, values θ and $\theta + 2\pi$ are identical.) Show that the free energy of this field is

$$\mathcal{F}[\theta(\mathbf{r})] = \text{const} + \frac{\hbar^2|\psi_0|^2}{2m}\int(\nabla\theta)^2\,\mathrm{d}V.$$

A uniform solution $\theta(\mathbf{r}) = \text{const}$ gives the absolute minimum of $\mathcal{F}$. (This theory also describes long-wavelength properties of magnets with an easy-plane anisotropy.)

(c) Soliton excitations of this model are *vortex lines*. The phase θ advances through $2\pi m$ as we travel around a vortex, where m is an integer. For a vortex line on the z axis, $\theta(\mathbf{r}) = \theta_0 + m\phi$, where ϕ is the azimuthal angle in cylindrical coordinates $\mathbf{r} = (\rho\cos\phi, \rho\sin\phi, z)$. Show that such a solution is indeed an extremum of the free energy $\mathcal{F}[\theta(\mathbf{r})]$. (It is, in fact, a local minimum.)

(d) Compute the free energy of a vortex. You will most likely encounter a divergent expression, which is directly caused by a lack of a length scale (like the correlation length ξ) in a model with a continuous broken symmetry.

(e) Nature, of course, abhors singularities and will do its best to avoid them. Any ideas how it will fight the divergences in this case?

4.[1] **Renormalization group treatment of the Ising chain.** Follow the derivation in Chapters 13.2A and 13.4A and fill in the gaps (Prob. 13.2) in Pathria.

[1] This problem is *optional.* It will not be graded and thus has no deadline. However, I strongly recommend it if you are planning to take 171.704.