Advanced Statistical Mechanics 171.703

Homework Assignment 2 (corrected)

Due date Friday, February 13

Reading. Chapters 9,11–16 in Landau.

Problem 1. Ideal gas. Consider an ideal gas of N nonrelativistic of particles with mass m occupying a volume V. The gas is in thermal equilibrium with kinetic energy E.

(a) Compute the number of states $\Gamma(E)$ whose energy does not exceed E. From that, deduce the entropy $S = \log \Delta\Gamma$ as a function of energy E. (The answer depends somewhat on the width of the energy distribution ΔE; however, the entropy *per particle* S/N is insensitive to such details when N is a macroscopic number.)

(b) Determine the absolute temperature T from the dependence of S on E. Determine the energy E at given T, N, and V.

Problem 2. Paramagnet. Consider $N \gg 1$ noninteracting spins $S = 1/2$ in an applied magnetic field $\mathbf{B} = (0, 0, B)$. The energy spectrum is discrete:

$$E = -\mu_B \, B \sum_{n=1}^{N} \sigma_n = -\mu_B \, B(N_\uparrow - N_\downarrow),$$

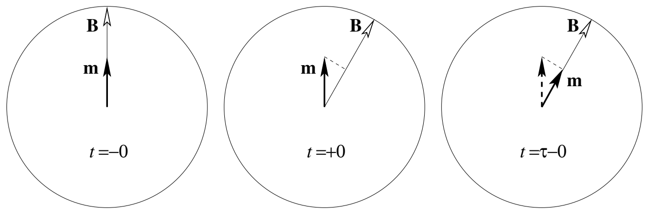
where $\sigma_n = \pm 1$, μ_B is the Bohr magneton, $N_\uparrow$ and $N_\downarrow$ are the numbers of spins pointing up and down. Work in the microcanonical ensemble.

(a) Determine the number of quantum states $\Delta\Gamma$ for an energy level with a given $N_\uparrow$ and its entropy $S = \log \Delta\Gamma$.

(b) Compute variations of energy ΔE and entropy ΔS when $N_\uparrow$ is increased by 1 (at constant N). Thus determine the absolute temperature T as a function of $N_\uparrow$ and $N_\downarrow$.

(c) Note that T can be *negative* for this system. (This happens when $N_\downarrow > N_\uparrow$.) Suppose you put in thermal contact two paramagnets with negative temperatures $T_1 < T_2 < 0$. Which way will the heat flow?

(d) Determine the energy E at given T, N, and B.



Problem 3. Paramagnet in a slowly rotating field. Slow rotation of the magnetic field $\mathbf{B} = B(\sin\theta, 0, \cos\theta)$ is an example of an adiabatic process. The spin Hamiltonian is now time-dependent:

$$H = -\mu_B\, \mathbf{B}(t) \cdot \boldsymbol{\sigma},$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the 2×2 Pauli matrices. The density matrix of the ensemble ρ can be written in terms of Bloch's polarization vector $\mathbf{m}$:

$$\rho = (1 + \mathbf{m} \cdot \boldsymbol{\sigma})/2.$$

The length of the vector $\mathbf{m}$ determines the fraction of the spins pointing along the field: $m = (N_+ - N_-)/(N_+ + N_-)$. In equilibrium, $[H, \rho] = 0$. If the direction of the field is suddenly changed, $[H, \rho] \neq 0$ and the density matrix evolves in such a way that the longitudinal (with respect to $\mathbf{B}$) component of $\mathbf{m}$ stays constant, whereas the transverse components decay on a time scale τ. Our task will be to show that

$$\mathrm{d}S/\mathrm{d}\theta = A\, \mathrm{d}\theta/\mathrm{d}t, \qquad A > 0, \tag{1}$$

which means that sufficiently slow rotation is adiabatic: $S = \text{const}$.

(a) Verify that the quantum-mechanical average for the entropy $S = -\mathrm{Tr}\,(\rho \log \rho)$ per spin agrees with the result obtained in Prob. 2(a) for a large ensemble $N \gg 1$. Use the large-n expansion $n! \approx (n/e)^n \sqrt{2\pi n}$.

(b) Show that, for a weak polarization, the entropy is $S \approx \log 2 - m^2/2$.

(c) In a rough model of transverse relaxation in rotating field, we represent continuous evolution of the density matrix with discrete steps of duration τ. First the field is instanteneously rotated through the angle $\Delta\theta = \dot{\theta}\tau$ in the xz plane. Subsequently the transverse component of polarization decays to zero, restoring thermal equilibrium. Determine the rate at which the entropy increases and thereby prove Eq. 1.