Advanced Statistical Mechanics 171.703

Homework Assignment 3

Due date Friday, February 20

Reading. Chapters 28, 31, 32 in Landau; 3.3, 3.4, 3.7 in Pathria.

1. Specific heat of classical oscillators. Determine the specific heat of harmonic and anharmonic oscillators with the Hamiltonians

$$H_1 = \frac{\omega(p^2+q^2)}{2}, \quad H_2 = \frac{\omega(p^2+q^4)}{2}, \quad H_3 = \frac{\omega(p^4+q^4)}{2},$$

(a) by using the identities valid in a canonical ensemble:

$$\left\langle q\,\frac{\partial H}{\partial q}\right\rangle = \left\langle p\,\frac{\partial H}{\partial p}\right\rangle = T;$$

(b) by direct evaluation of the partition function.

2. Paramagnet. Derive thermal properties of a $S = 1/2$ paramagnet with the Hamiltonian

$$H = -\mu_B\,\mathbf{H}\cdot\sum_{n=1}^{N}\boldsymbol{\sigma}_n.$$

Determine the partition function Z, free energy $F(T,H) = -T\log Z$, specific heat, magnetization $M = -(\partial F/\partial H)_T$, and isothermal susceptibility $\chi = (\partial M/\partial H)_T$.

3. Scaling for power-law interactions. Consider a gas of classical particles whose interactions are a homogeneous function of n-th degree:

$$V(\lambda\mathbf{r}_1,\ldots,\lambda\mathbf{r}_N) = \lambda^n\,V(\mathbf{r}_1,\ldots,\mathbf{r}_N).$$

The absence of a characteristic distance in the Hamiltonian $H = K(p)+V(q)$ leads to a *scaling behavior* of various physical observables. Mathematically, scale invariance limits the partition function to this form (Landau, p. 93):

$$Z(V,T) = T^{3N(1/2+1/n)}\,f(VT^{-3/n}). \tag{1}$$

Use Eq. 1 to establish the following properties.

(a) Show that different isotherms (graphs of P vs V at constant T) have the same shape. In other words, upon proper rescaling of P and V, all isotherms collapse into a single curve.

(b) Derive the result obtained in class via the virial theorem:

$$E + \frac{3}{n}PV = 3NT\left(\frac{1}{2} + \frac{1}{n}\right).$$

4. Thermodynamic perturbation theory. Problem 3.29 in Pathria.