Advanced Statistical Mechanics 171.703

Homework Assignment 8

Due date Friday, April 9

Reading. Chapters 12.1, 12.3 in Pathria.

Simulations. Download and play with Michael Creutz's simulator `xpotts`. The Potts model is a generalization of the Ising model in which spin variables take on q different values. You get the Ising model by setting $q = 2$. The simulator, along with other `xtoys`, can be found at Creutz's web site thy.phy.bnl.gov/www/xtoys/xtoys.html. See the file xpotts.txt for download and installation instructions. Perform suggested experiments.

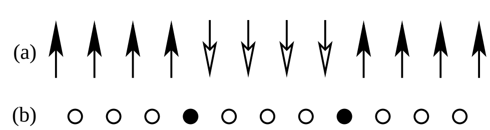
1. Ising domain walls as ideal fermions. The Ising ferromagnet on a chain has the energy

$$H = -J \sum_i \sigma_i \sigma_{i+1} - h \sum_i \sigma_i,$$

where $\sigma_i = \pm 1$ is a spin variable on site i. An alternative way to compute the partition function and spin correlations in the 1d Ising model is to use domain walls, a.k.a. *kinks*, to parametrize configurations of the system. A kink is a boundary between two arrays of spins up and down [Fig. (b)]. It lives on links, rather than sites, of the chain [Fig. (b)]. In the 1d Ising models kinks behave as ideal fermions because:

- there can either be 0 or 1 kinks on any link;
- the energy of a kink is $2J$, whether or not there is another kink nearby.

(a) Compute the partition function of a chain with L sites and *open* boundary conditions.



(b) Compute the spin-spin correlation $\langle \sigma_0 \sigma_n \rangle$ as a function of n. *Hint:* The product $\sigma_0 \sigma_n$ equals $(-1)^k$, where k is the number of kinks between sites 0 and n.

(c) Show that the correlation function decays exponentially with the distance: $\langle \sigma_0 \sigma_n \rangle = \exp(-n/\xi)$. Compute the correlation length ξ as a function of temperature.

2. Magnetic susceptibility of the 1d Ising ferromagnet. Define the magnetic moment M and magnetization m as follows:

$$M = \sum_{i=1}^{L} \sigma_i, \qquad m = \langle M \rangle / L.$$

(a) Relate the spin susceptibility $\chi = \lim_{h \to 0} (\partial m / \partial h)$ to the fluctuation of the magnetic moment: $\chi = \langle M^2 \rangle / LT$. (This is not specific to the Ising model but is a rather general thermodynamic identity.)

(b) Compute the spin susceptibility of the Ising chain as a function of temperature.

3. Magnet with infinite-range interactions. One of the simplest models exhibiting a phase transition is the Ising ferromagnet with infinite-range interactions: every spin interacts with every other spin:

$$H = -\frac{J}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j,$$

where $\sigma_i = \pm 1$ is again an Ising variable. The interaction constant is normalized by N in order to make the energy an extensive, $\mathcal{O}(N)$ quantity.

(a) Determine energy levels $\{E_n\}$ and their degeneracies $\{d_n\}$.

(b) Show that, for large N, the partition function can be approximated as follows:

$$Z = \sum_n d_n \exp(-E_n/T) \approx C(N) \int_{-1}^{1} \mathrm{d}m \, \exp(-NU(m,T))$$

where $m = (N_\uparrow - N_\downarrow)/N$ is the magnetization variable.

(c) In the limit $N \to \infty$, the probability density $\rho(m,T) = C \exp(-NU(m,T))$ is narrowly peaked at the value $m = m_0$ minimizing the potential $U(m,T)$, so that m *always* equals m_0 and fluctuations are negligibly small. Obtain the equation of state relating m_0 and T.

(d) Determine the critical temperature T_c separating the high-temperature paramagnetic phase ($m = m_0 = 0$) from the low-temperature ferromagnetic phase ($m = \pm m_0 \neq 0$). Sketch the dependence $m(T)$.