Advanced Statistical Mechanics 171.703

Final Exam

Prof. Tchernyshyov

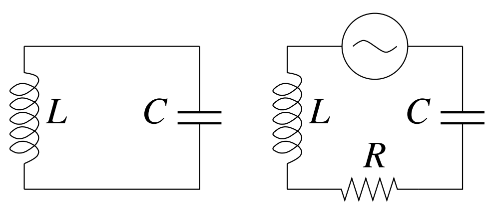
6 May 2004

- You may use notes, books, and calculators.
- You may NOT consult with anyone.
- Return the exam by 9 am on Friday, 7 May, to 323 Bloomberg.

1. Density fluctuations: fermions. In a small (but still macroscopic) fraction of an enclosed volume, the number of particles N does not remain constant. We have computed its fluctuation for a *classical* ideal gas:

$$\Delta N \equiv \left\langle (N - \langle N \rangle)^2 \right\rangle^{1/2} = \langle N \rangle^{1/2}.$$

Compute ΔN for a degenerate Fermi gas.



(a) (b)

2. Nyquist noise. Determine the r.m.s. amplitude of the noise voltage $U_{\text{rms}} = \langle U^2 \rangle^{1/2}$ across a capacitor $C = 1$ nF at room temperature. The inductance of the coil is $L = 1$ mH.

(a) Do so for the idealized dissipationless circuit (in essence, a harmonic oscillator whose coordinate is charge Q).

(b) Add dissipation in the form of a finite series resistance R, compute the impedance of the circuit $Z(\omega)$ and demonstrate, with the aid of the fluctuation-dissipation theorem, that the noise amplitude U_{rms} remains the same.

3. Spin dimers. In some antiferromagnets, spins $S = 1/2$ are arranged in pairs. To the first approximation, such *dimers* can be considered independently of each other. The Hamiltonian of a dimer in magnetic field $\mathbf{B} = (0, 0, B)$ is

$$H = J\,\mathbf{S}_1 \cdot \mathbf{S}_2 + 2\mu_B B(S_1^z + S_2^z),$$

where $J > 0$ is the exchange constant and μ_B is the Bohr magneton. The energy levels are shown in the table below. (S_{12} is the total spin of the dimer and S_{12}^z is its projection on the direction of $\mathbf{B}$.)

S_{12}	S_{12}^z	E
	$+1$	$J/4 + 2\mu_B B$
1	0	$J/4$
	-1	$J/4 - 2\mu_B B$
0	0	$-3J/4$

The magnetic field is weak: $\mu_B B$ is small in comparison to both J and $k_B T$.

(a) Compute the free energy of a dimer in zero field. Determine the entropy and specific heat.

(b) Compute the magnetic susceptibility of a dimer

$$\chi = \lim_{B \to 0} \langle \mu \rangle / B,$$

where $\mu = -2\mu_B S_{12}^z$ is the magnetic moment.