# Non-affine Elasticity in Jammed Systems

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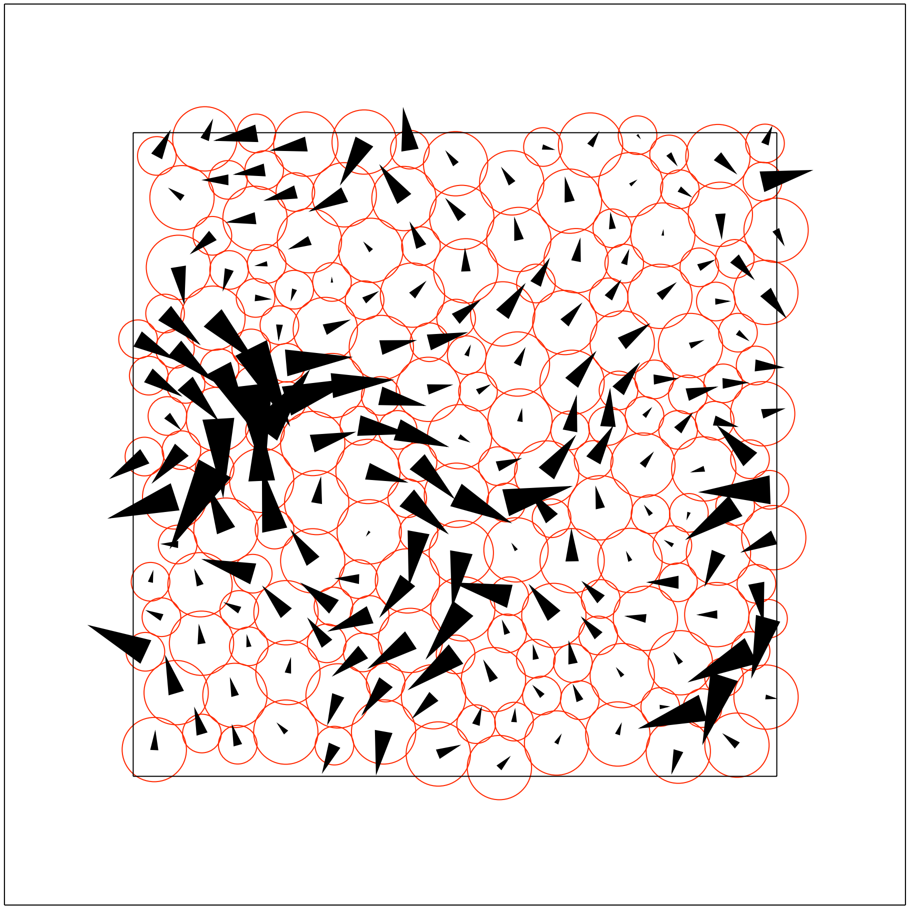
APS March Meeting
2006
Baltimore, MD

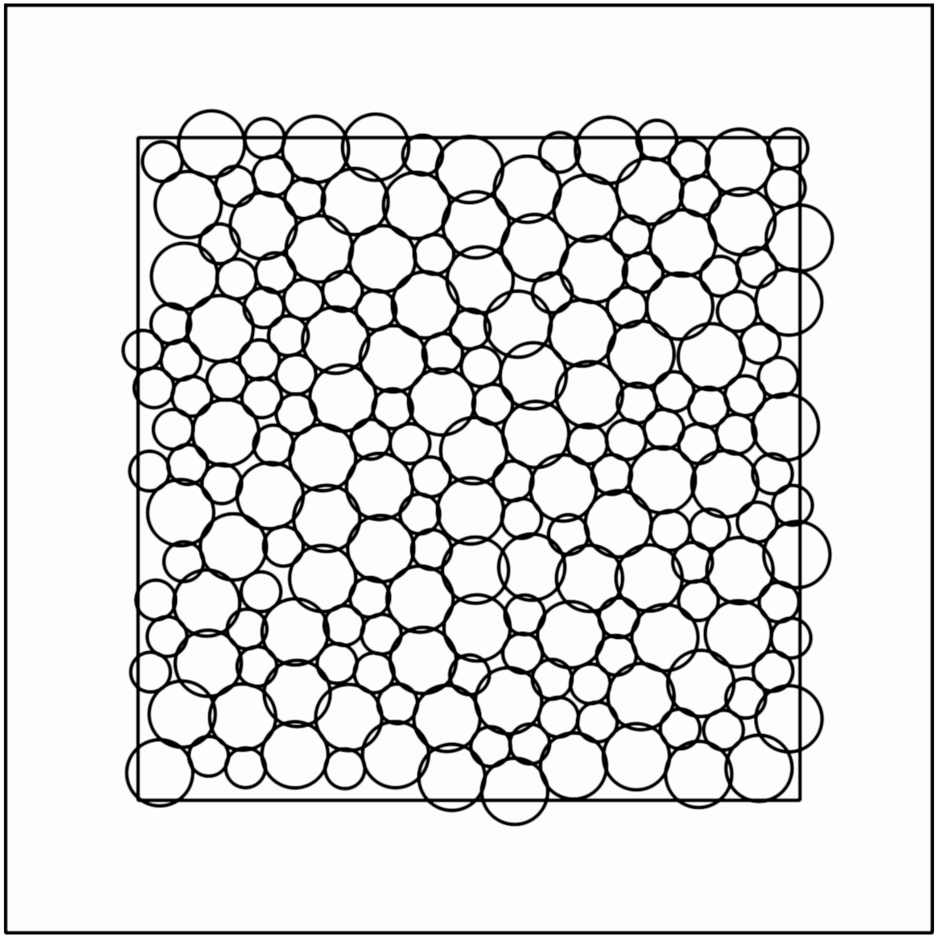


# Acknowledgments


- A. Lemaître, Institut Navier
- J.S. Langer, UC Santa Barbara
- M.O. Robbins, JHU
- Funding: LLNL University Relations

# Non-affine Response

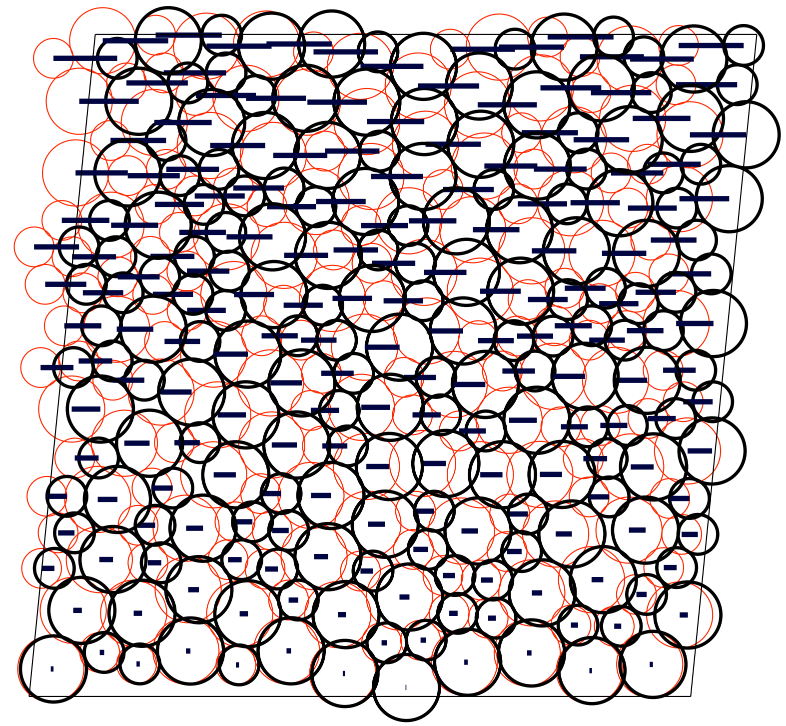
- Sequence:
  - Initial packing, F=0



- What is this stuff?
  - Bubbles or
  - Grains or
  - Atoms

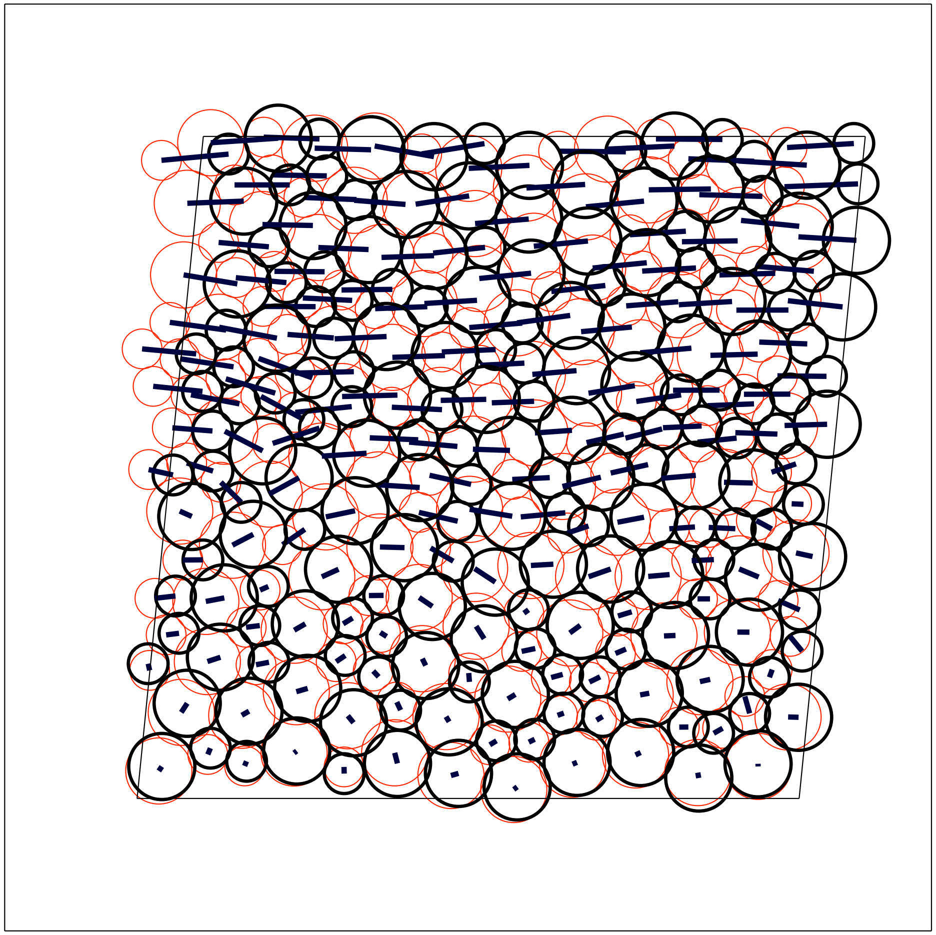
# Non-affine Response

- Sequence:
  - Initial packing, F=0
  - Sheared state, F!=0



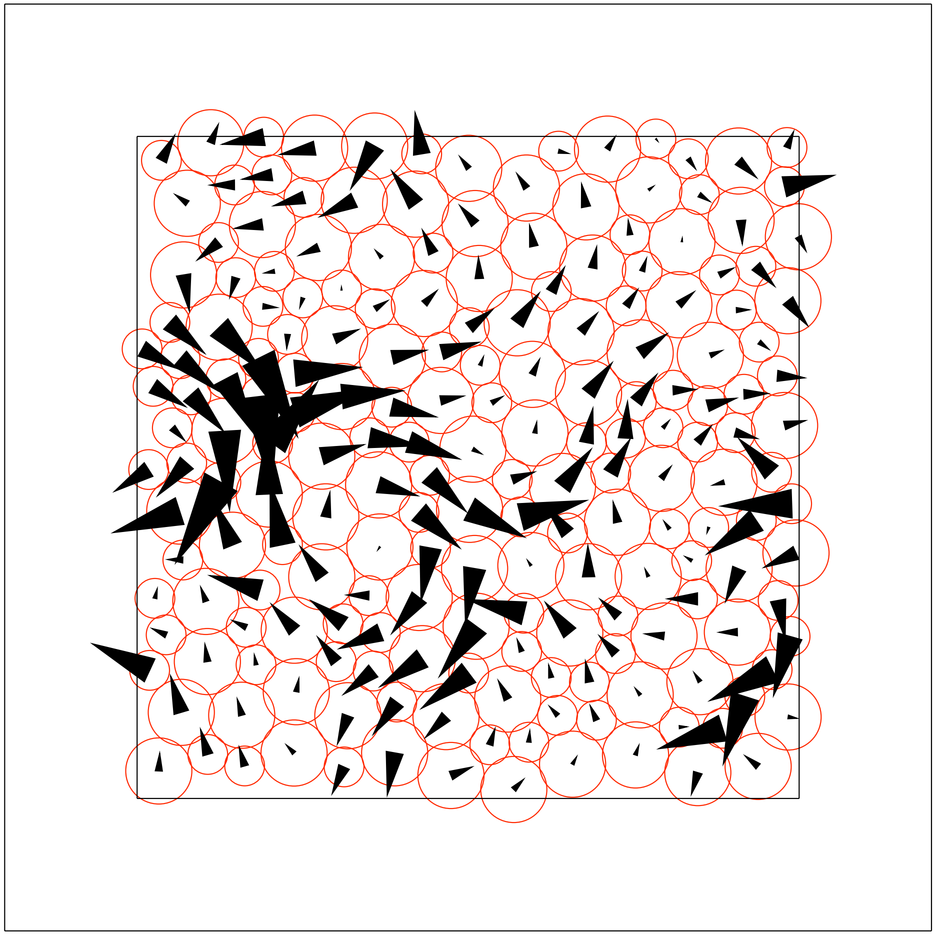
# Non-affine Response

- Sequence:
  - Initial packing, F=0
  - Sheared state, F!=0
  - Allow correction so F=0 again.



# Non-affine Response

- Sequence:
  - Initial packing, F=0
  - Sheared state, F!=0
  - Allow correction so F=0 again.
  - Subtract affine piece.

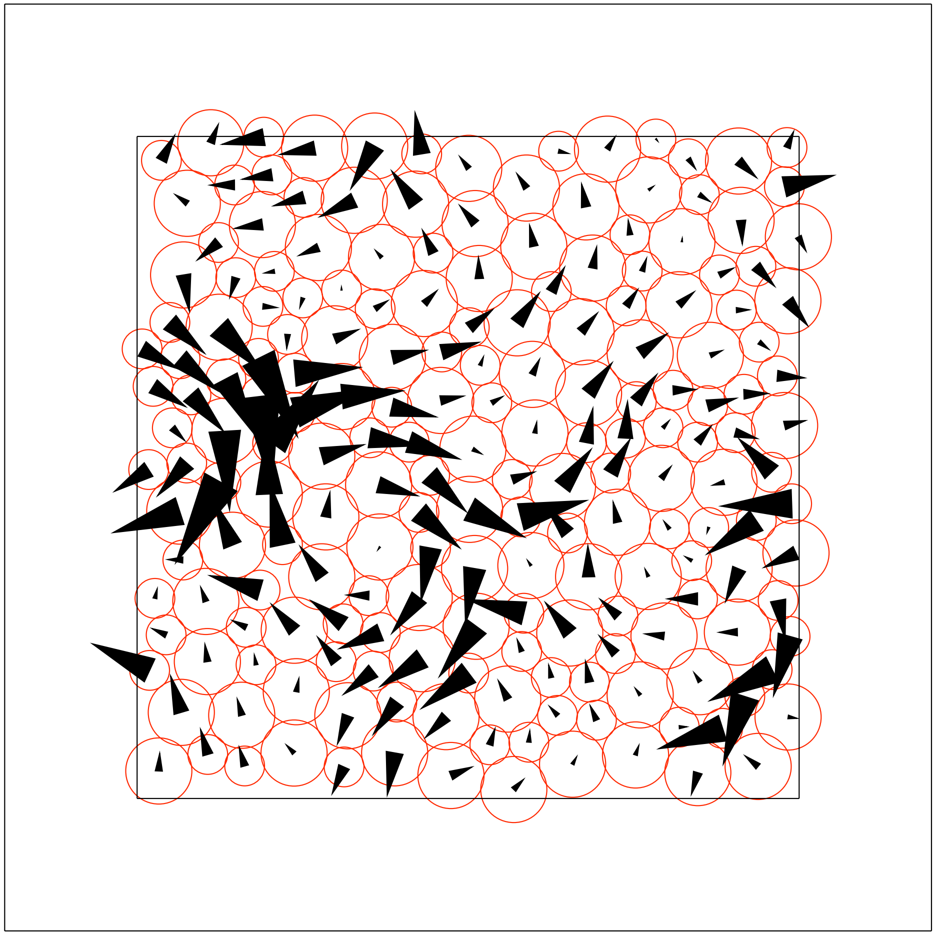


# Motivation:

Q) How to characterize the local disorder?
A) The “affine forces”, $\Xi$

Q) Can a characteristic length be defined?
A) No. Vortices scale with system size.

Q) How do heterogeneities in the elasticity initiate plasticity?
A) Elastic response localizes into a shear zone



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- Leonforte, et. al., find a characteristic vortex size.
- DiDonna and Lubensky develop a framework which exhibits log divergences.
- We develop a similar framework, but conclude that vortices are scale free.
- Can get good quantitative agreement with the data.

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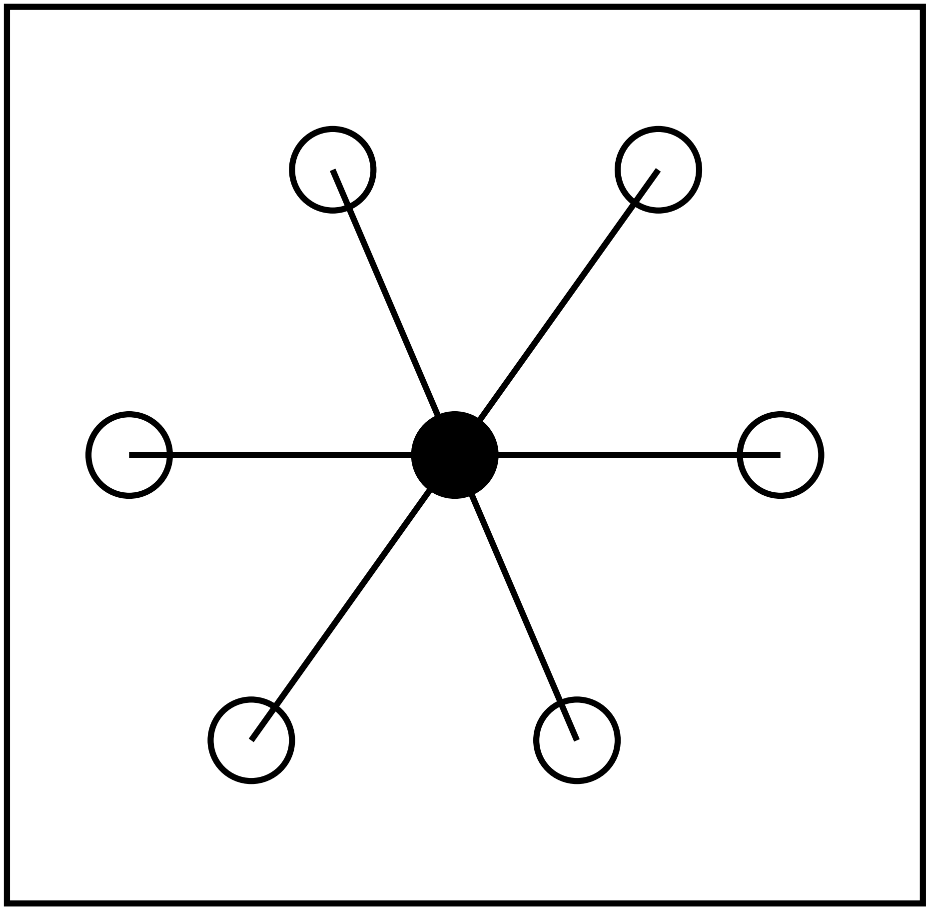
- Older studies [Srolovitz et. al. Acta Metal. 1981] find that plasticity is nucleated near stress concentrations.
- In our systems, plasticity is instead nucleated at regions of large non-affine elasticity.
- We derive analytical expressions for this nucleation process.

# Computing the response

- Single particle toy problem:
  - Start at F=0

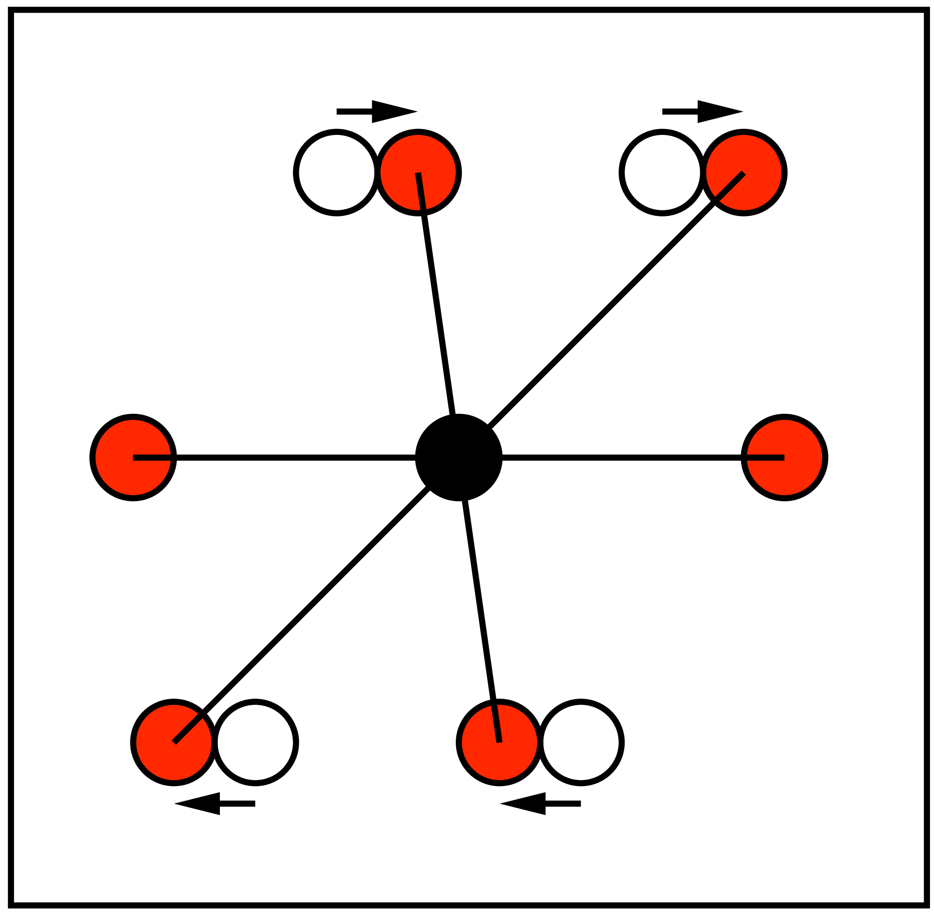
Ordered Case



# Computing the response

- Single particle toy problem:
  - Start at F=0
  - Apply affine shear
  - Forces remain zero
  - No correction necessary

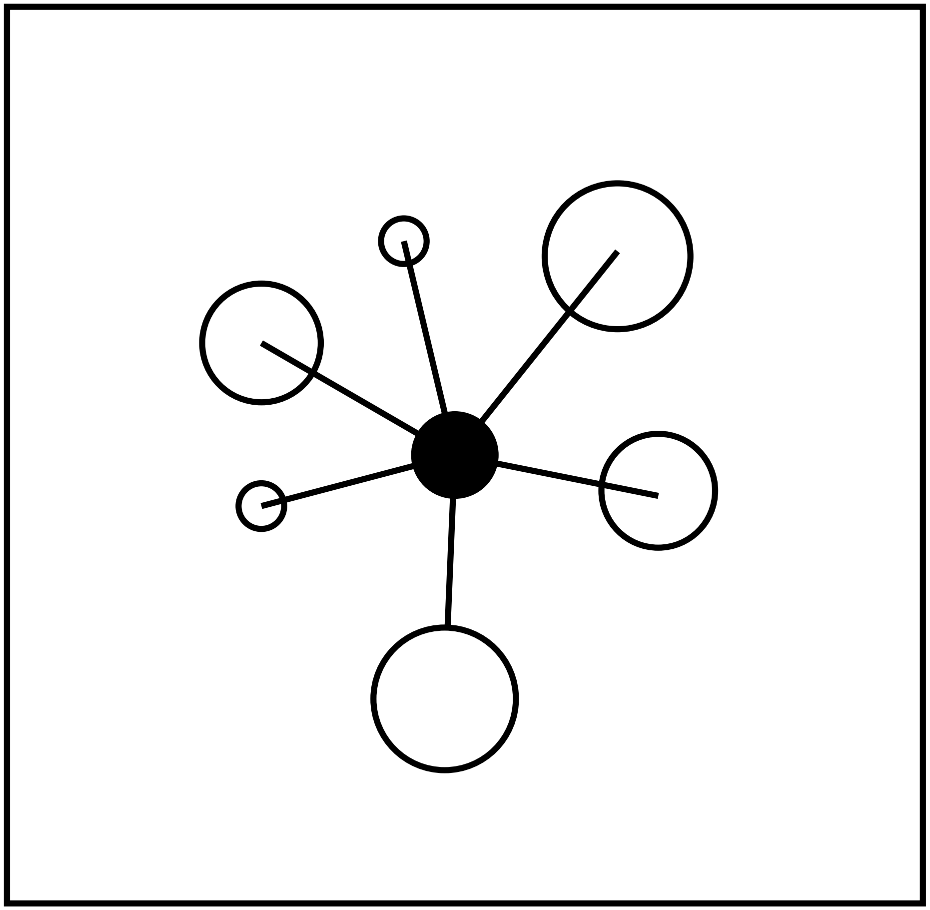
## Ordered Case



# Computing the response

- Single particle toy problem:
  - Start at F=0

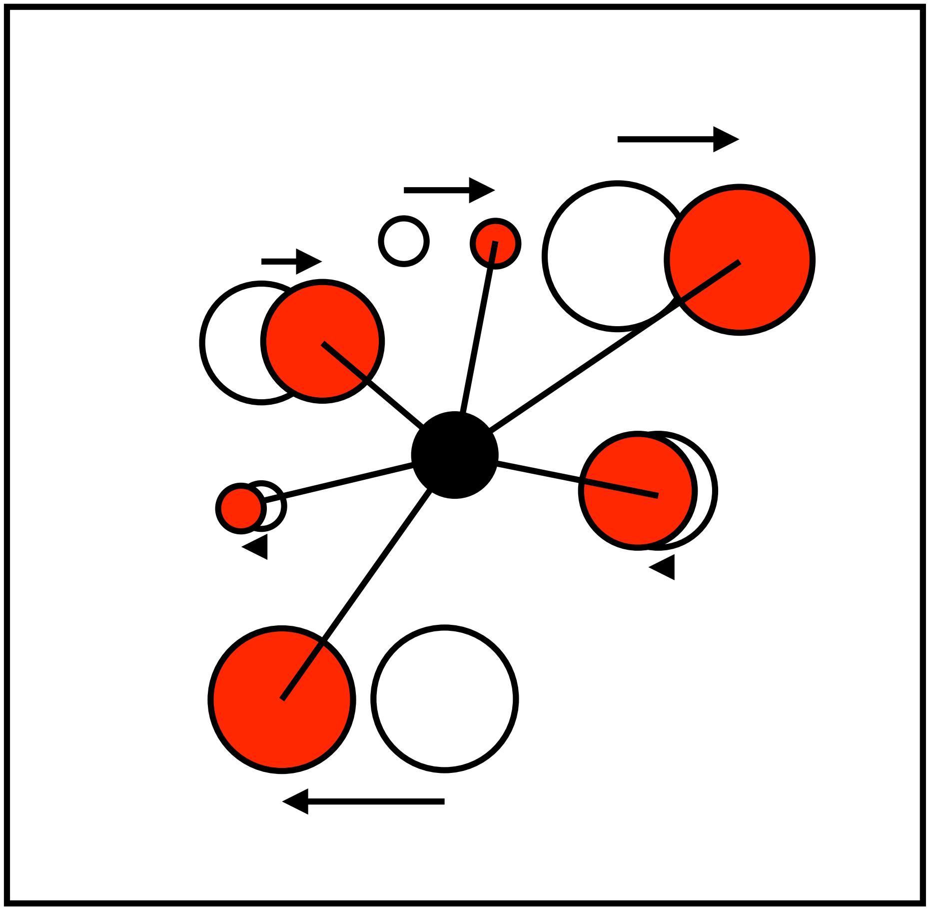
Disordered Case



# Computing the response

- Single particle toy problem:
  - Start at F=0
  - Apply strain

Disordered Case



# Computing the response

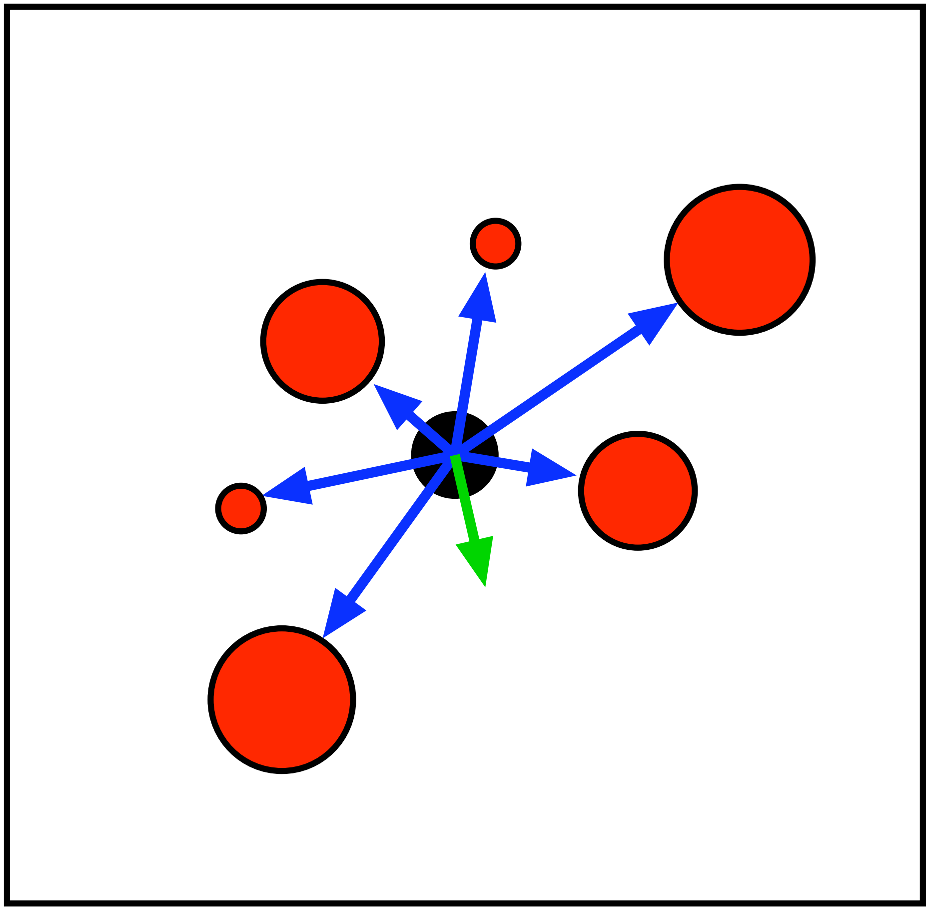
- Single particle toy problem:
  - Start at F=0
  - Apply strain

Use Hessian to compute "Affine force"

$$\vec{\Xi}_i = \sum_j \mathbf{H}_{ij} \vec{dr}_j$$

$$\vec{\Xi}_i = \gamma \sum_j \mathbf{H}_{ij} \hat{\mathbf{x}} \delta y_j$$

Disordered Case



# Computing the response

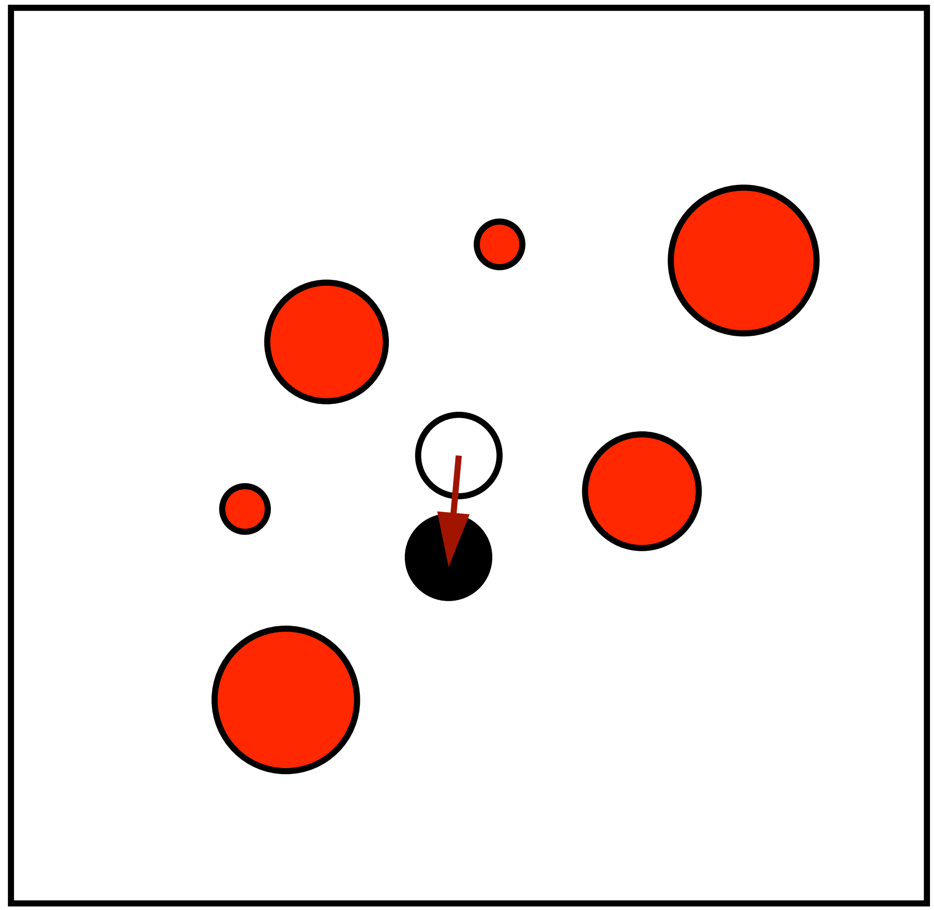
- Single particle toy problem:
  - Start at F=0
  - Apply strain

Use Hessian to find position correction

$$\vec{\Xi}_i = \mathbf{H}_{ii} \vec{dr}_i$$

$$\vec{dr}_i = \mathbf{H}_{ii}^{-1} \vec{\Xi}_i$$

Disordered Case

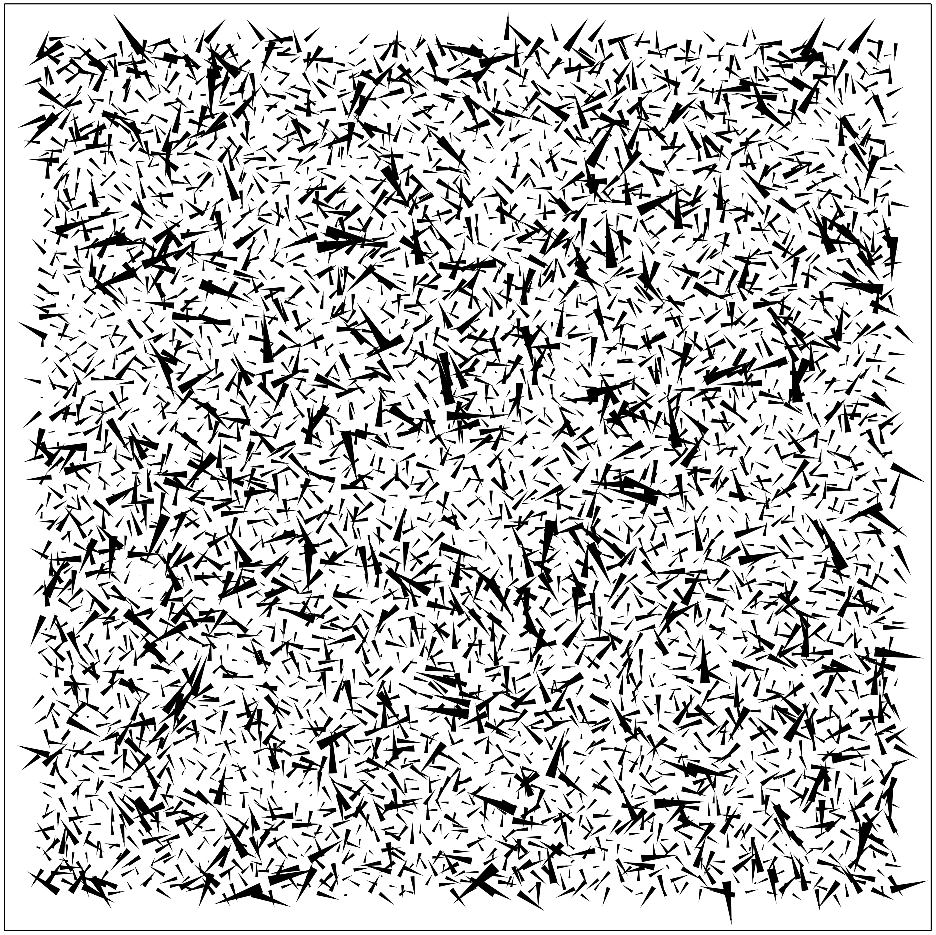


# Computing the response

- Back to full assembly:

$$\vec{\Xi}_i = \gamma \sum_j \mathbf{H_{ij}} \hat{\mathbf{x}} \delta y_{ij}$$

  - Measure of local disorder.
  - No spatial correlations in our samples.

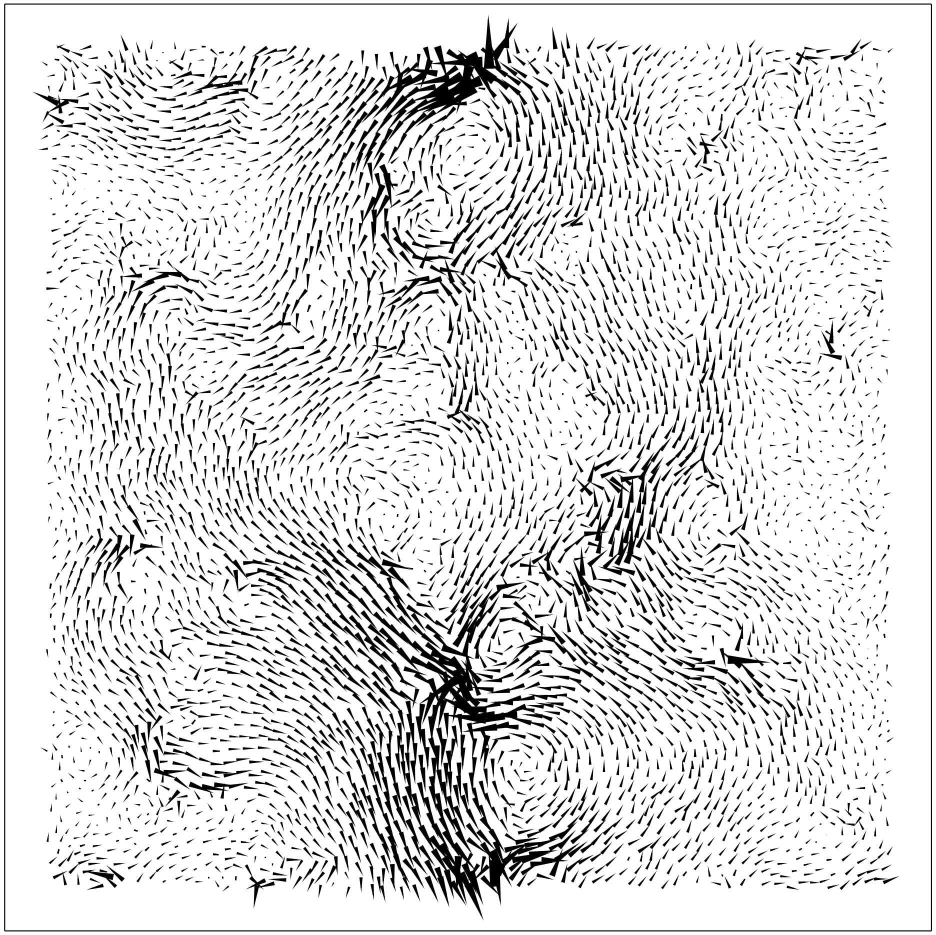


# Computing the response

- Back to full assembly:

$$\vec{dr}_i = \gamma \sum_j \mathbf{H}_{\mathbf{ij}}^{-1} \vec{\Xi}_j$$

Force balance:
Affine forces, $\Xi$, must be balanced by correction forces, $H^{-1}_{ij} dx_j$



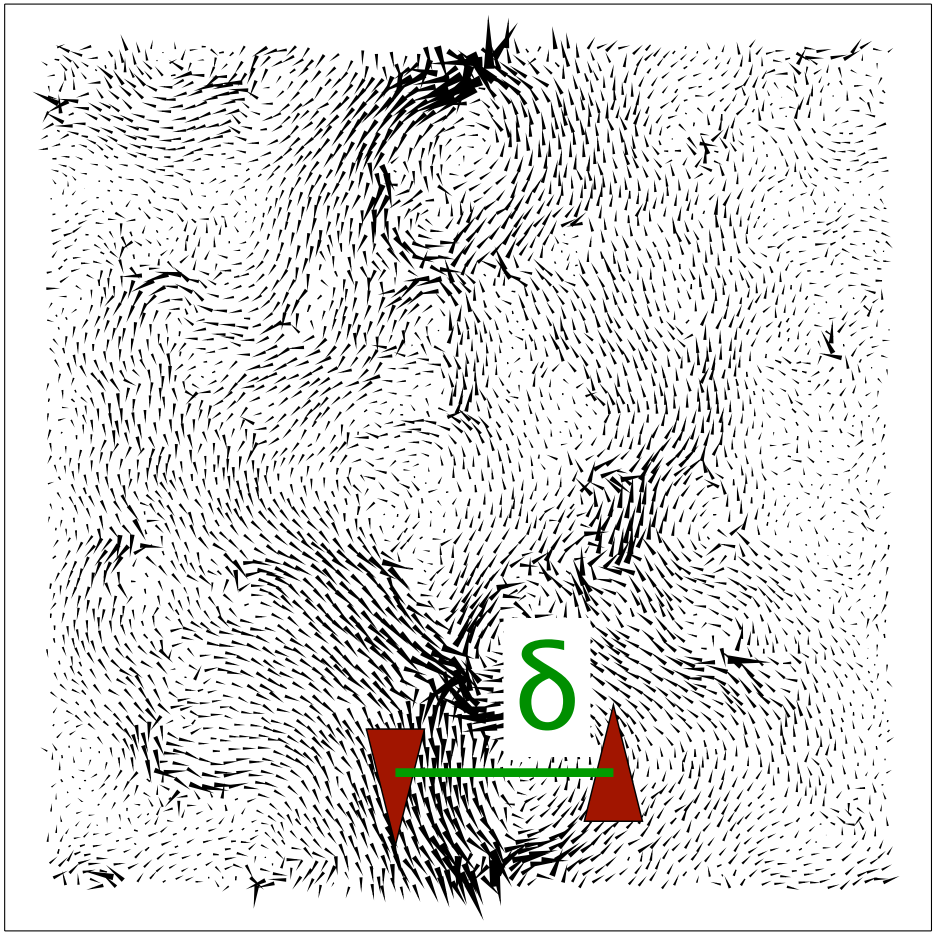
# Outline

- Overview
- Scale free vortices: (CEM [PRL submitted?])
    - Autocorrelation g(r)
    - Normal-mode decomposition
- Plastic nucleation
- Outlook

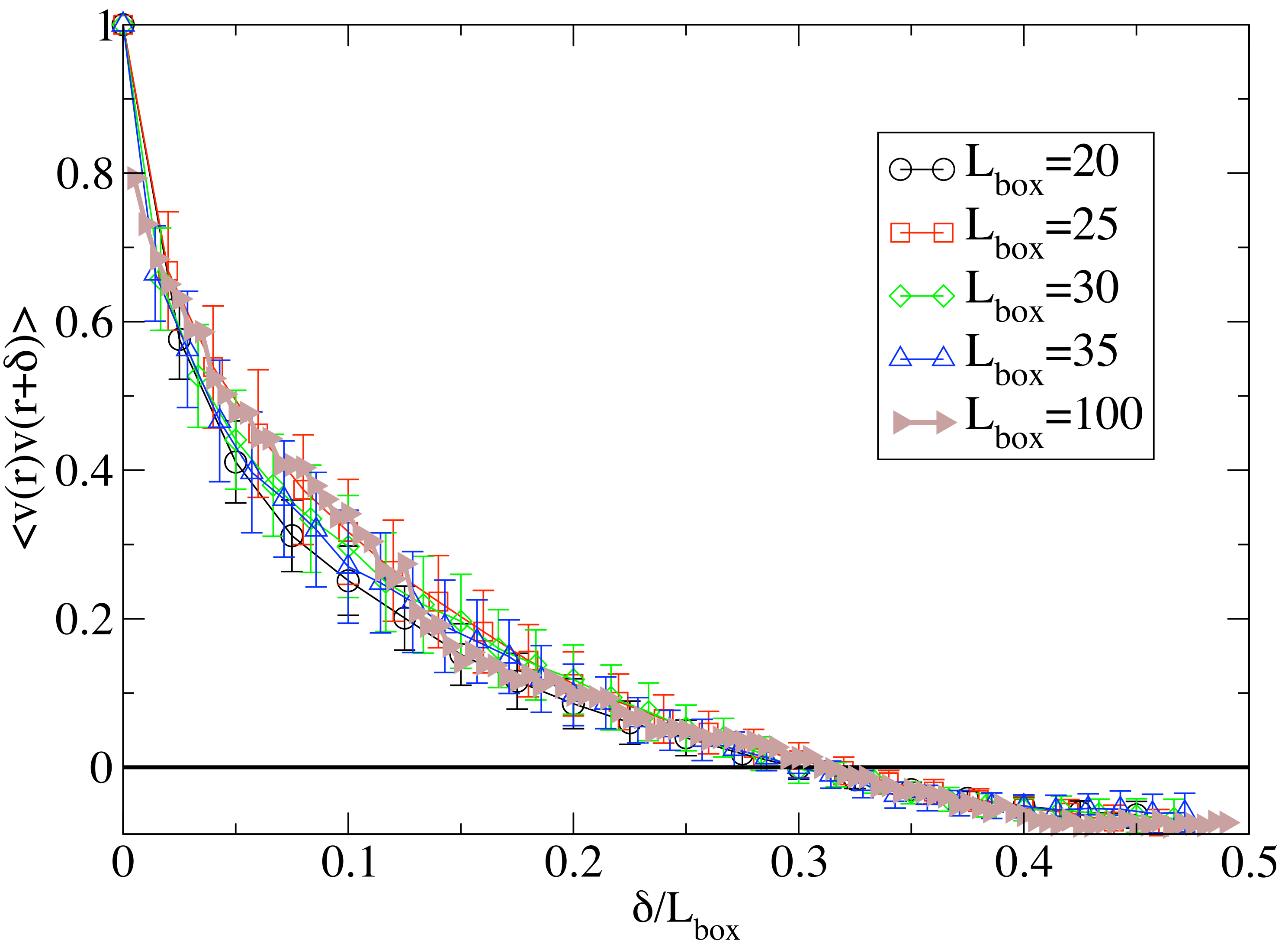
# Autocorrelation, g(δ)

$$g(\vec{\delta}) \doteq \int \vec{v}(\vec{r}) \cdot \vec{v}(\vec{r} + \vec{\delta}) d\vec{r}$$

- Usual autocorrelation
- Measures "vortex size"
- Characteristic length?



# Autocorrelation, g(δ)



# g(δ): Theoretical form

Recall:

$$\vec{dr}_i = \gamma \sum_j \mathbf{H_{ij}^{-1}} \vec{\Xi}_j$$

Then:

$$\vec{dr}_i = \gamma \sum_p \left( \frac{\Xi_p}{\lambda_p} \right) \vec{\psi}_{ip}$$

- Note:
  - $\Xi_p$ are random
  - $\Psi_p$ are plane waves to first order in $\Xi$

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- Note:
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Approximate $dr_i$ as random sum of plane waves:

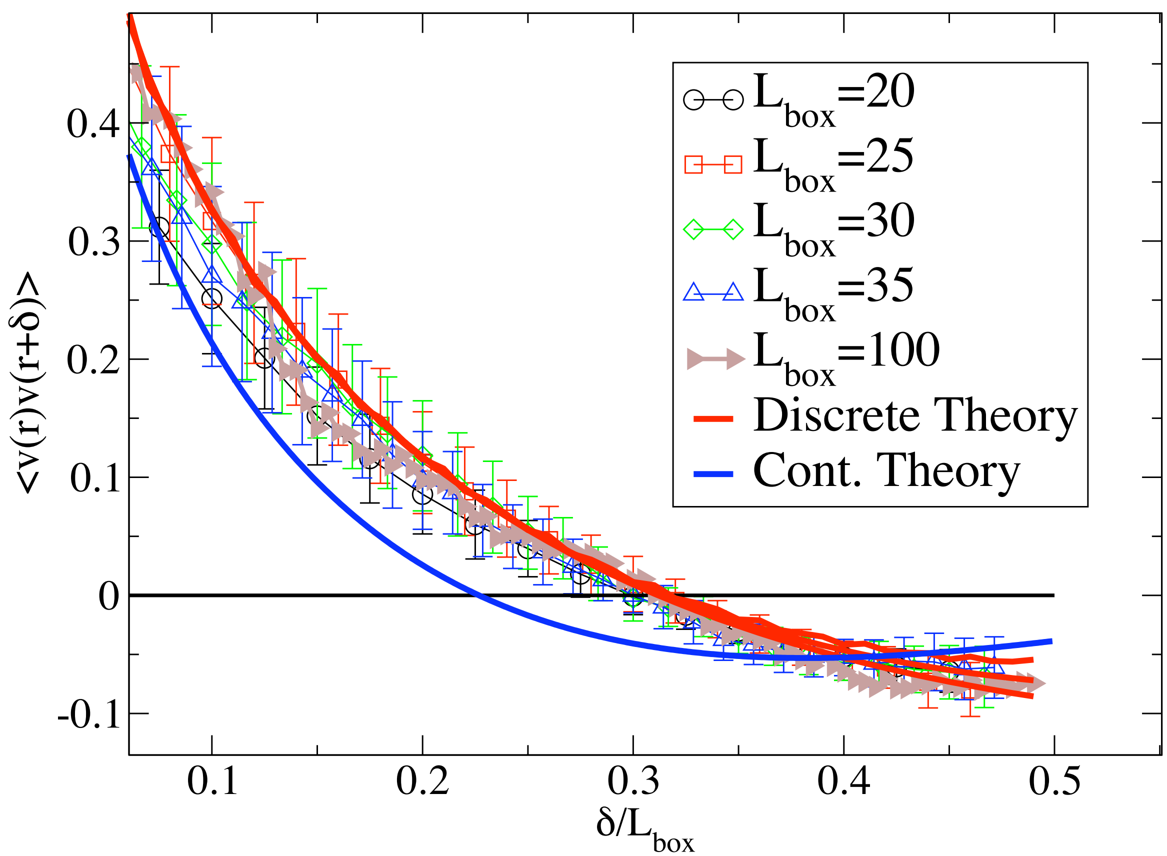
$$\vec{dr}_i \sim \sum_{k=(m,n)} \phi_{mn} \frac{e^{2\pi i \vec{k}\cdot\vec{x}_i/L}}{|\vec{k}|}$$

Then g(δ) is:

$$g(\vec{\delta}) \sim \sum_{k=(m,n)} \frac{\cos(2\pi\vec{k}\cdot\vec{\delta}/L)}{k^2}$$

# Simulation and Theory

$$g(\vec{\delta}) \sim \sum_{k=(m,n)} \frac{\cos(2\pi\vec{k}\cdot\vec{\delta}/L)}{k^2}$$




Similar to DiDonna +Lubenksy,

- $g(k) \sim 1/k^2$

but:

- Fully discrete derivation

Blue curve:
Semi-continuum

Red curve(s):
Partial sum (n=40)
3 different angles

# Outlook

Summary:

- Displacement field from random forces on a homogeneous sheet.
- Predicts "vortex length" ~ .32 $L_{box}$
- No length scale comes out of data or theory.

Future Direction:

- When does the assumption of uncorrelated $\Xi$ break down?
- Can this bring out a characteristic length?
- How to make systematic pert. expansion for H?

# Outline

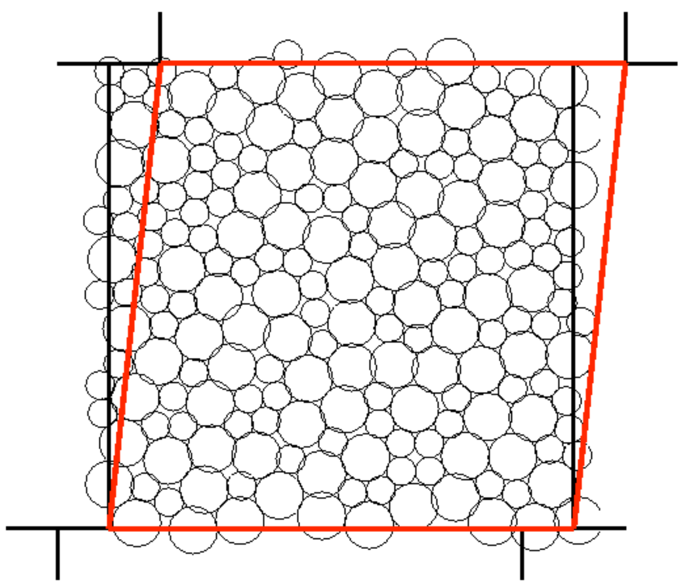
- Overview
- Scale free vortices
- Plastic nucleation (CEM+A. Lemaître [PRL 2004] [PRE submitted])
  - Energy Landscape Perspective
  - Breakdown of Elasticity in Real Space
- Outlook

Only if I have time.

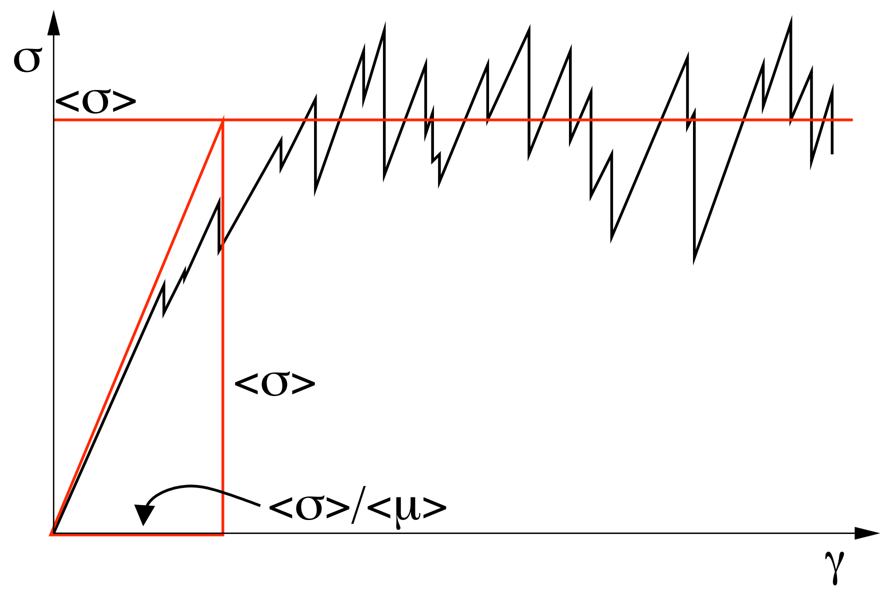
# Large Strains

- ...l:
  - •Minimize energy
  - •Shear system
  - •Repeat

- Procedure is:
  - Athermal, Quasi-static
  - “minimalist”



“Lees-Edwards” Cell

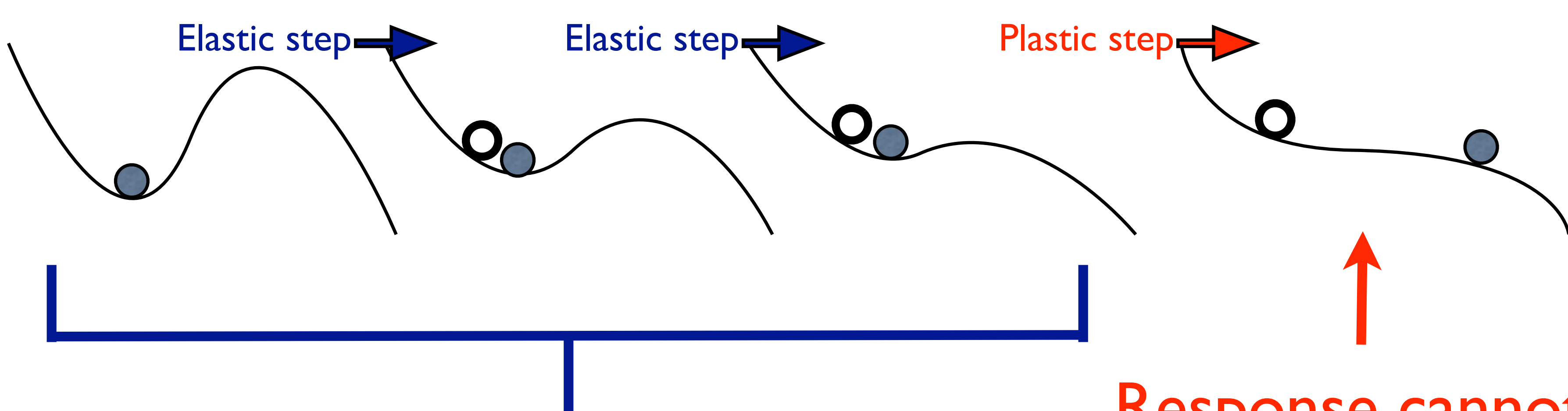



Typical Stress-Strain Curve

# Landscape Perspective

Increasing strain ⇨

After Malandro
and Lacks

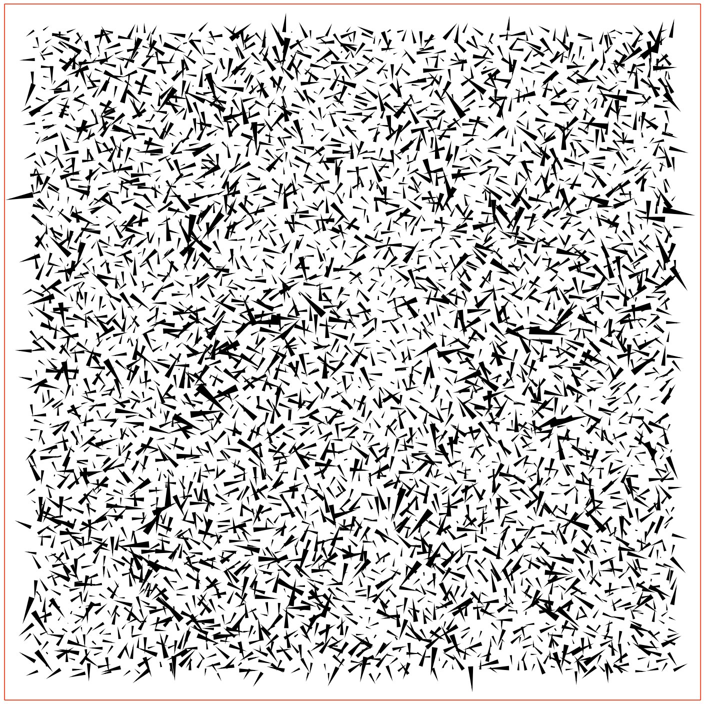



Response can be linearized.
Deformation is reversible (elastic).

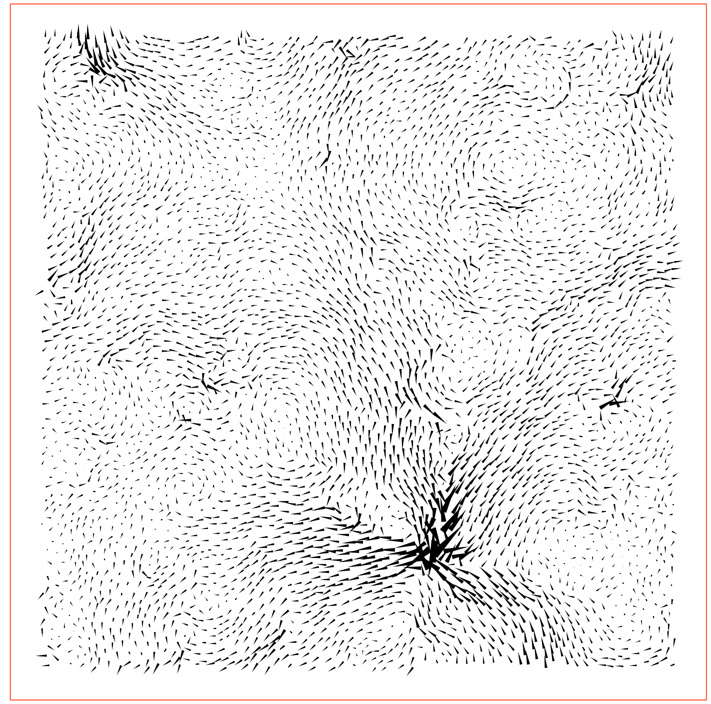
Response cannot be linearized. Deformation is irreversible (plastic).

Recall: $\vec{dr}_i = \gamma \sum_j \mathbf{H_{ij}^{-1}} \vec{\Xi}_j$

# Singular Mode



$\Xi$



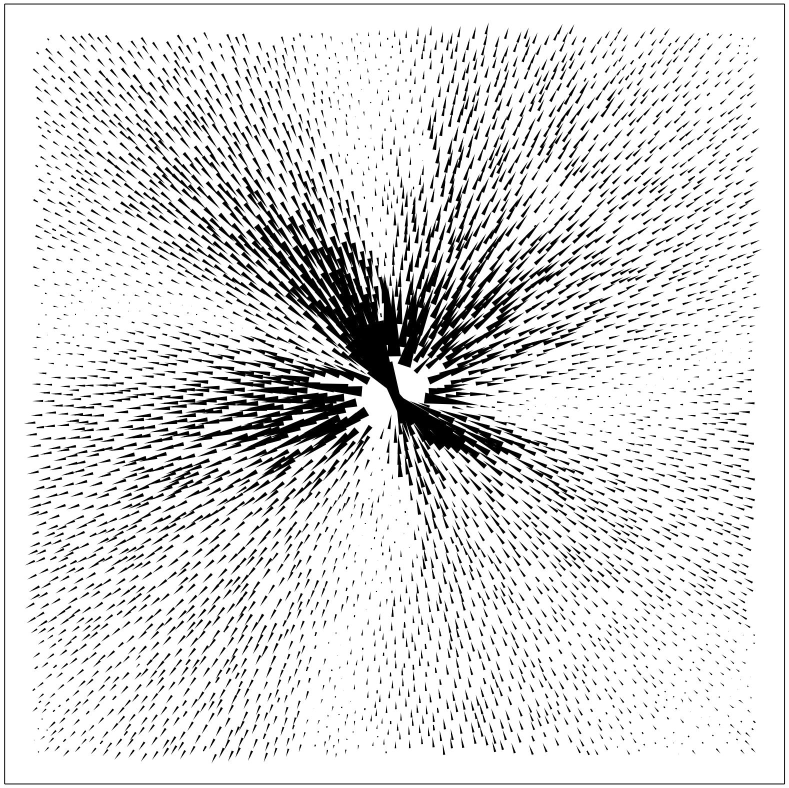
dr

Plastic nucleation is intrinsically non-local!

Cannot be detected via $\Xi$!

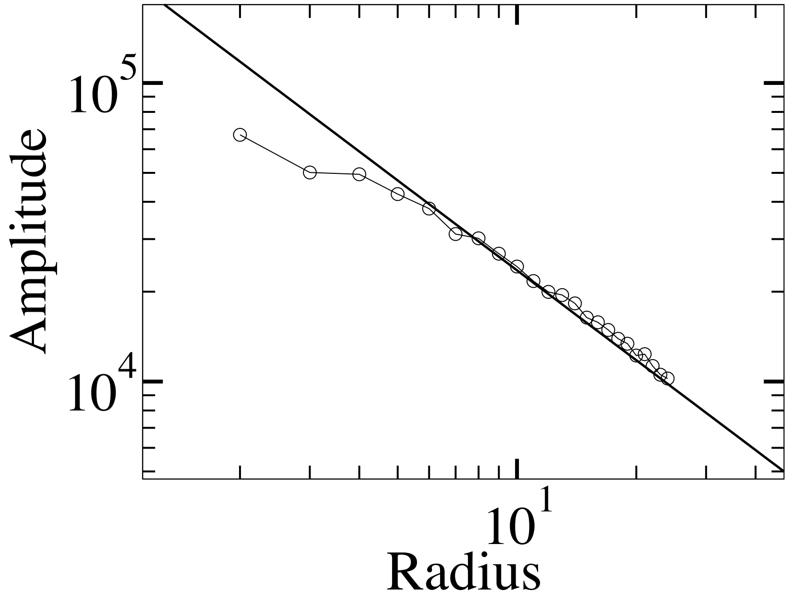
# Singular Mode

Can critical mode be rationalized elastically?



Lamé-Navier predicts, for quadrupoles:

$$v_r(r) = \frac{2A}{r^3} + \frac{(1+\kappa)B}{r}$$



# Outlook

Summary:

- Diverging elastic displacement triggers plastic nucleation
- Onset of plasticity is NOT detectable via the local quantities ($\sigma$,$\Xi$,$\mu_{Born}$, etc)

Future Direction:

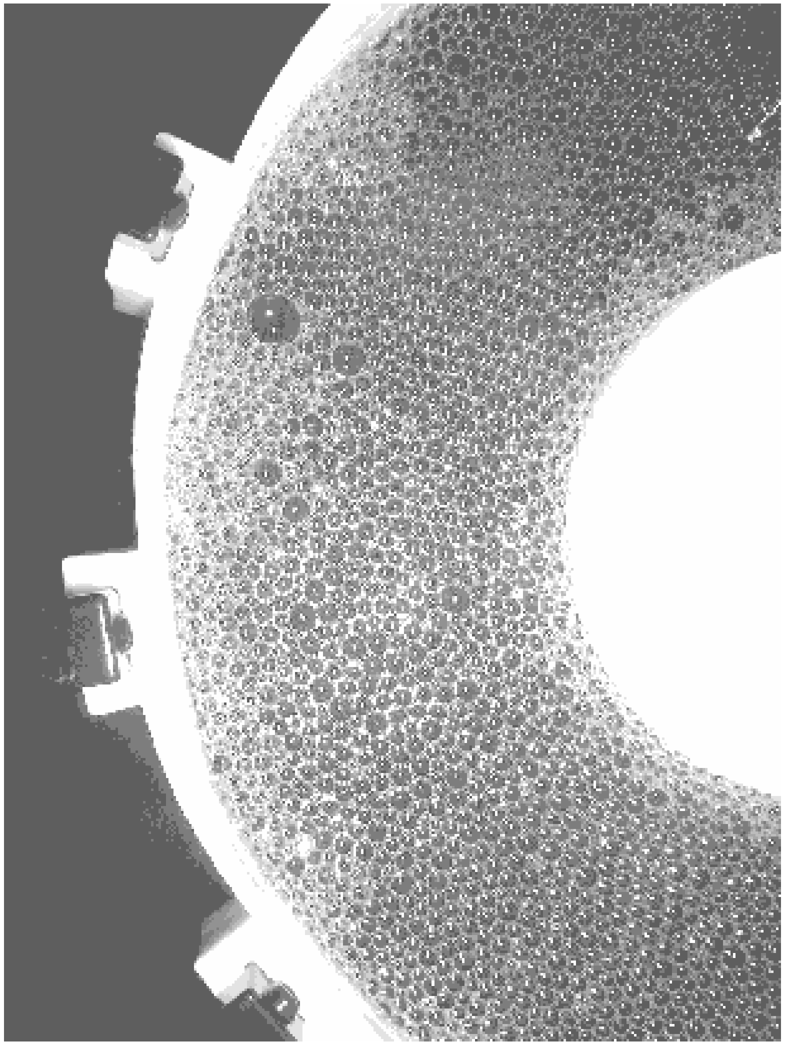
- Can a critical "core" region be defined?
- How might these core regions affect the non-critical elastic behavior?

# Related APS talks:

- Majumdar (Behringer) B8
- Schall (Cohen, Spaepen, Weitz) B21
- Abate, D1
- Walder (Dennin, Levine) R8

# Jammed Systems

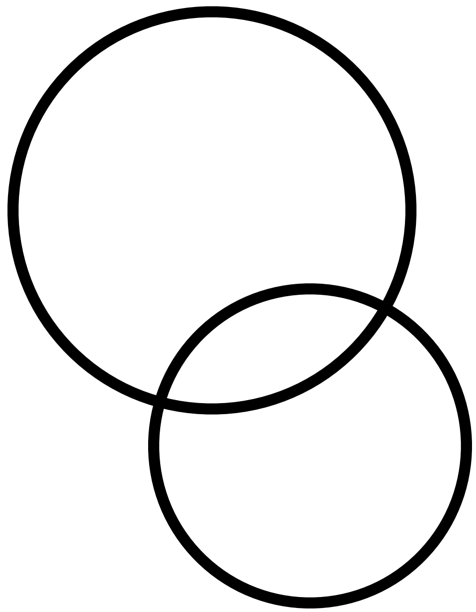
- Examples:
  - Bubbles/Emulsions
  - Grains
  - Glasses
- Non examples:
  - Suspensions / Rigid Grains
- Differences:
  - Inertia/Temp/Dissipation
- Similarity:
  - Geometry!
- Issues:
  - Characterizing disorder
  - Elasticity / Vibrations
  - Plasticity / Yielding



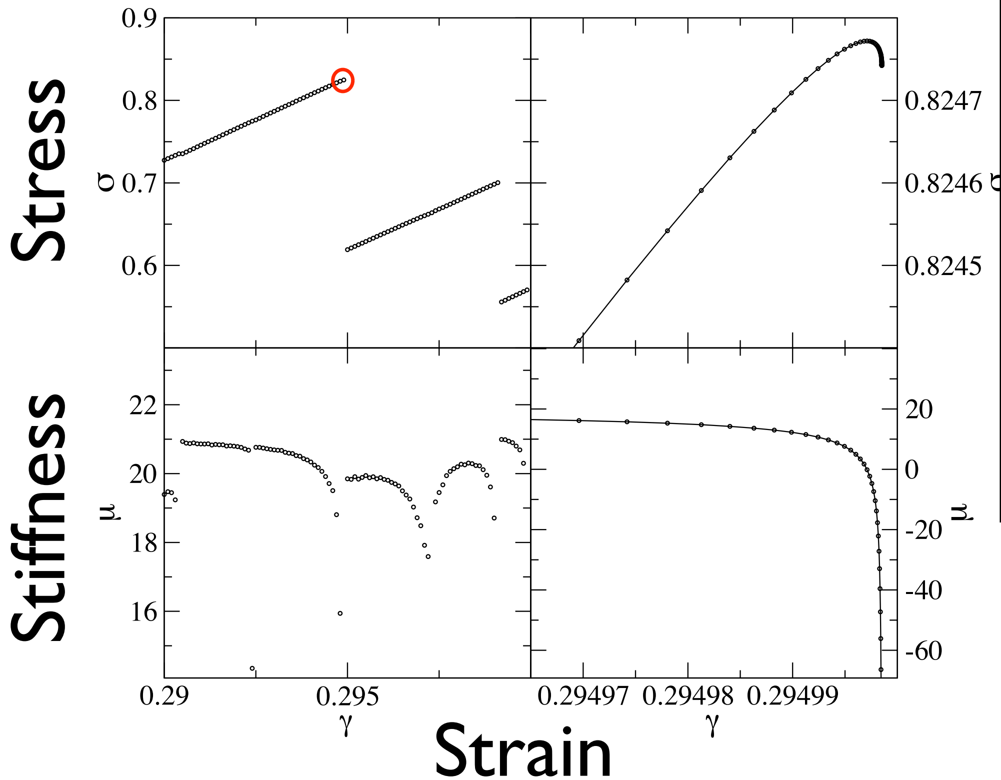
From (M Dennin)

# Numerical protocol

- All results for 2D
- Binary mixtures to prevent crystalization
- Interactions:
  - Harmonic contact repulsion
  - Standard Lennard-Jones 6-12
- Preparation: "violent" quench from initial random state.



# Approach to Singularity




Initiation of single plastic event

$$\frac{d\sigma}{d\gamma} = \frac{\partial\sigma}{\partial\gamma} + \sum_i \frac{\partial\sigma}{\partial r_{i\alpha}} \frac{dr_{i\alpha}}{d\gamma}$$

$$= \frac{\partial\sigma}{\partial\gamma} - \sum_{ij} \Xi_{i\alpha} H^{-1}_{i\alpha j\beta} \Xi_{j\beta}$$

$$= \frac{\partial\sigma}{\partial\gamma} - \sum_p \frac{\Xi_p^2}{\lambda_p}$$

Catastrophe Theory:

$$\lambda_0 \sim \sqrt{\delta\gamma}$$

$$\mu \sim -(\delta\gamma)^{-1/2}$$