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An Introduction to Supersymmetry

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Abstract

This is a writeup of a lecture given for second-year graduate student seminar in Physics and Astronomy at the Johns Hopkins University. The basics of supersymmetry are developed in a self-contained way that assumes no prior knowledge of field theory or particle physics, aside from topics discussed earlier in other presentations. After covering the necessary notions in field theory (Lagrangians, Klein-Gordon, spinors, symmetries and gauge fields), we approach SUSY from basic examples and develop the topic of superfields and superspace and gauge transformations, after which we apply these ideas to supersymmetric quantum electrodynamics and the minimal supersymmetric extension of the Standard Model. Some important consequences of SUSY are discussed, including the famous hierarchy problem of Higgs scalar mass.

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I. INTRODUCTION AND OVERVIEW

Modern theoretical physics is rooted in the study of symmetry, that is, the study of the transformations we can perform on a system and get "the same physics". The central concepts of freshman physics, those of conservation of energy and linear and angular momentum, we find arise from the invariance of physical law under time and space translation and spatial rotation, respectively. All fundamentally true theories of physics, particularly at high energies, must be relativistically invariant, and so must be symmetric under the group of rotations between space and time that leave invariant the length of a vector in Minkowsky spacetime. The particle content of the Standard Model is defined by the vector space that transforms under the symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$. The purpose of this lecture is to present a new, speculative symmetry that has gained a very prominent and fashionble status among possible new physics to be sought out at the LHC. This is supersymmetry("SUSY"): a fundamental symmetry in nature between bosons and fermions, and the goal of the present work is to present how it works, its mathematical foundations and why it is so important.

As this is a talk to be given to a wide audience of physics graduate students, I have assumed no specialized knowledge of particle physics or field theory, so the first part will be to put across basic specific background material that will be of use later on. Basic ideas of Lagrangian densities and second quantization are presented first, then the relativistic Klein-Gordon equation for scalar particles, then spinors defined as complex representations of the Lorenz group, leading to the Dirac Lagrangian. We discuss actual physical systems with interactions using Quantum Electrodynamics. This same model is used to motivate general gauge symmetries. The other background material is given part-way through the lecture and concerns the Standard Model and electroweak spontaneous symmetry breaking via the Higgs mechanism in particular.

This presentation came out of self-study from a number of sources that I would like to cite at the beginning. There are several books and lecture notes devoted to introducing supersymmetry. Probably the standard is Wess and Bagger's Supersymmetry and Supergravity[1], but another helpful source is the text by West [2], and a less-well-known book that does everything in pain-staking but often illuminating and positively-reinforcing detail is the text by Mueller-Kirsten and Wiedemann[7]. There is a well-known supersymetry primer by Martin[6], which forms the basis of our discussion of the simplest supersymmetric Lagrangians as well as the interaction in the Minimally Supersymmetric Standard Model (MSSM) and possible mechanisms explaining why SUSY as we observe it is broken, and the lecture notes by Tata[9] are used here to motivate SUSY breaking, and are an excellent source for possible experimental details that I cannot touch on here. Most of our discussion of the MSSM, however, follows the TASI lectures on weak-scale supersymmetry by Prof. Bagger[8], as does our discussion of superfields and supersymmetric gauge fields.

Our overview is as follows. After the aforementioned crash course on the essentials of QFT we find the simplest supersymmetric Lagrangians through the Wess-Zumino model, or rather the equivalent one that uses Weyl fermions(as done in [6],[8]) rather than the original Majorana ones([9],[2],[7]). From this, we construct supersymmetry as a group of transformations acting on the internal components of a field, from which we introduce superfields transforming in superspace, and discuss interesting consequences of the algebraic relations of the generators *vis-a-vis* "all possible symmetries of the S-Matrix". In addition to its philosophical virtue(treating SUSY as just another group of symmetries), it makes writing an interaction Lagrangian whose action is invariant under SUSY an easy task, and allows us to introduce gauge fields in an elegant way(see [1], Chap. 6,7). From this, after discussing particularly relevant aspects of the Standard Model, we put forth its minimal supersymmetric extension. This sets the stage for the important predictions of the MSSM, most famously a solution to the hierarchy problem, as well as R-parity and a potential dark matter candidate and the unification of all three Standard Model gauge couplings at high energy.

Supersymmetry demands that each particle and its partner have the same mass, and the fact that these partners for SM particles have not been observed means that SUSY is broken in the universe we observe, and a complete discussion of even the basics of SUSY include models for its spontaneous breaking, but time constraints did not allow for any discussion. I simply can only refer the reader first to Chapter 8 of Wess and Bagger and Chapters 4,6,and 7 of [6], and Section 3.3-4 of [9]. The point of this talk is to get across what SUSY is and why it is important, and although experimental details should be brought up, here they are for the most part neglected, but I refer the reader to Tata's aforementioned lectures.

II. BASIC BACKGROUND MATERIAL: QUANTUM FIELDS AND GAUGE GROUPS

In this section, we cover the conceptual basics of field theory using the simplest quantum field theory, QED, as a useful example. In the standard undergraduate quantum mechanics, electromagnetic fields are treated semiclassically in terms of classical potentials that interact with the wavefunctions of our particles through the Schroedinger equation(i.e., we describe the hydrogen atom by $V(r) = -e^2/r$ and leave it at that). However, we also remember that the earliest discovery in quantum theory was the particle nature of light, and this has to go along with the wave/particle duality of electrons, so, to abuse classical terminology, we need a description of photons as waves behaving as particles behaving as waves, and the same description needs to apply to electrons in any fundamental picture of nature. Electrodynamic phenomena are described in a classical theory of fields, and we seek a quantum description of these fields, just as how we used single particle dynamics to describe an electron in quantum mechanics. Electrons and other single-particle states must be described in the same formalism as the photon for our system to be consistent and fundamental. So, by analyzing how classical fields behave, we arrive at second quantization, and get a description of all quantum states as a space acted on by a field operator. This material, and much of modern particle physics can be found in Peskin and Schroeder [3],Mandl and Shaw [10], and Zee's introduction [11]. The Standard Model is explained very clearly in Griffiths and in Halzen and Martin [4, 5].

A. Second Quantization and The Klein-Gordon Equation

Single particle classical mechanics is centered around an action$S$ defined by the Lagrangian $L = T - V$, dependent on a particle's generalized coordinates and tangeant vectors $\{(q_i, \dot{q}_i)\}$.

$$S = \int dt L(\{q_i, \dot{q}_i\}).$$

As usual, the system obeys the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0, \forall i.$$

If we have many interacting particles in a system, say N particles on a lattice, indexed by $\{\alpha = 1, ..., N\}$, our action is

$$S = \int dt L(\{q_i^\alpha, \dot{q}_i^\alpha\}).$$

If there are an infinte number of such particles, so that the lattice spacing approaches zero, and there is a particle everywhere, our coordinates are no longer indexed discretely, but as continuous functions of the space containing our system:

$$S = \int dt L(\{q_i(\vec{x}), \dot{q}_i(\vec{x})\}),$$

and the system under consideration becomes a *field*; the generalized coordinates go from describing particle trajectories to describing field amplitudes, which we rename ϕ. We can rewrite this action in terms of a function defined at each coordinate $\vec{x}$

$$S = \int dt d^3\vec{x} \pounds(\phi_i(\vec{x}), \partial_\mu \phi_i(\vec{x})),$$

where $\pounds(\phi_i(\vec{x}), \dot{\phi}_i(\vec{x}))$ is known as the Lagrangian density, herafter referred to simply as the *Lagrangian*. This quantity describes classical fields just as fundamentally as L described systems with individual particles. Applying the same integration by parts to minimize this action, we find, given a Lagrangian, its *field equations*:

$$\frac{\partial \pounds}{\partial \phi_i} - \frac{\partial}{\partial x^\mu} \frac{\partial \pounds}{\partial(\partial_\mu \phi_i)} = 0, \forall i.$$

Note that i can range over any internal as well as space-time indices(in the latter case, these are the dymanical equations for components of *vector* fields). Note also that in this formalism, there is no difference between space and time, provided that our Lagrangian makes no distinction, and so our field theory is Lorenz invariant, which is good since we want to be able to describe photons, and quantizing electrodynamics demands we obey special relativity.

As a quick example, we have from Jackson that electrodynamics is described by the Lagrangian

$$\pounds_{EM} = -\frac{1}{8\pi} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu$$
$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$

where $A^\mu = (\phi, \vec{A})$ is the 4-vector potential, which we treat as our field amplitude and $j^\mu = (\rho, \vec{j})$ a current density, from which we recover $\partial^2 A^\mu = 4\pi j^\mu$ if we work in the Lorenz gauge(and, as elsewhere in the paper, Heaviside-Lorenz units where $c = 1$).

What we must do now is remake single particle quantum mechanics into this field language, and make the result be Lorenz invariant. At this point, we are still neglecting spin, and we will soon describe spinor fields. We have derived classical equations in terms of physical observables. The usual way to quantize a system is to rewrite the physical observables as Hermitian operators acting on a space of states. Here, however, the physical observables are field amplitudes. In single particle quantum mechanics, all particles are described by a wavefunction, which is itself a field amplitude, except now this amplitude is being treated as a physical operator. This is the notion known as *second quantization.* In other words, for a wavefunction ψ,

$$\psi(\vec{x}) = \langle \vec{x}|\psi\rangle =_{\psi} \langle 1|\hat{\psi}(\vec{x})|1\rangle_{\psi}.$$

Relativistic quantum particles of spin zero (scalars) are described by the well-known Klein-Gordon Equation,

$$(\partial^2 + m^2)\phi = 0,$$

in the language of single-particle quantum mechanics. We reinterpret ϕ as a field amplitude operator, and we see that this is the field equation from the Lagrangian

$$\pounds_{KG} = \partial^\mu \hat{\phi}^\dagger \partial_\mu \hat{\phi} - \hat{\phi}^\dagger m^2 \hat{\phi}.$$

This Lagrangian is given full justice in any book on quantum field theory, but time and spatial constraints prohibit a detailed discussion. What is important to notice is that the Klein-Gordon equation can be written in such a way that ϕ behaves like the dynamical variable of a simple harmonic oscillator:

$$\frac{\partial^2 \hat{\phi}}{\partial t^2} + (\vec{p}^2 + m^2)\hat{\phi} = 0,$$

with frequency $\omega = \sqrt{\vec{p}^2 + m^2}$. This indicates that $\hat{\phi}$ can be written as a combination of bosonic creation and annihilation operators:

$$\hat{\phi} = \frac{1}{\sqrt{2\omega}}(\hat{a}(\vec{p}) + \hat{a}^\dagger(\vec{p})).$$

These operators act on Hilbert space states $|n\rangle_\omega, n \in \mathbf{Z}$, for frequency ω. n is interpreted as the number of particles present with this frequency, and the operators have the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ that defines the harmonic oscillator. With this harmonic oscillator analogy, we can define our scalar field operators dependent on x as an inverse Fourier transform over momentum(i.e, an expansion in energy or frequency eigenstates) thusly

$$\hat{\phi}(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} (\hat{a}(\vec{p}) e^{-ipx} + \hat{a}^\dagger(\vec{p}) e^{ipx}).$$

This whole oscillator approach can be legitimized further by considering the momentum density canonically conjugate to $\hat{\phi}$, defined in analogy to how conjugate momenta are defined in single-particle Lagrangian mechanics.

$$\hat{\pi} = \frac{\partial \pounds}{\partial \dot{\hat{\phi}}} = \dot{\hat{\phi}}^\dagger.$$

$$\hat{\pi}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} i\sqrt{\frac{\omega_{\vec{p}}}{2}} (\hat{a}^\dagger(\vec{p}) e^{ipx} - \hat{a}(\vec{p}) e^{-ipx}),$$

and this gives us the bosonic commutation relation we would expect as a density:

$$[\hat{\phi}_\alpha(x), \hat{\pi}_\beta(y)] = i\delta_{\alpha\beta}\delta(x-y),$$

where we have defined α, β as discrete indices over some internal space, and the Kronecker delta comes from the form of the kinetic part of the Klein-Gordon equation, and we used the commutator of the raising and loweing operators defined in this sum

$$[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{q})] = (2\pi)^3 \delta(p-q).$$

Also, we can find the amplitude for a scalar field to move from spacetime point x to y (the Green's function or propagator):

$$D_F(x-y, m) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip(x-y)}}{p^2 - m^2},$$

defined by its Fourier transform. These are the basic concepts of scalar fields, and these are all derived in detail in any field theory text(see particularly Mandl and Shaw[**?**] and Peskin and Schroeder[3] for it done in this canonical way). We state without proof that something similar is done for photons, and this can be motivated by the fact that the four potential A^μ is like describing four scalars, so the difference with scalar fields (and constant factors) are the polarization degrees of freedom.

$$\hat{A}^\mu(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2V\omega_{\vec{p}}}} (\epsilon^\mu(\vec{p}) \hat{a}(\vec{p}) e^{-ipx} + \epsilon *^\mu (\vec{p}) \hat{a}^\dagger(\vec{p}) e^{ipx}).$$

B. Spinors and the Lorenz Group

In this section, we discuss how fermions behave in the field formalism, and how they transform under spacetime rotations. Scalar fields, as their name suggests, transform as Lorenz scalars. We decribe these kinds of fields under the usual cordinate transformations generated infinitessimally from the identity by writing

$$\phi'(x') = \phi(x)$$

$$\phi(x') = \phi(x) + \partial_\mu \phi \delta x^\mu \rightarrow \phi'(x) = \phi(x) - \partial_\mu \phi \delta x^\mu,$$

for a field ϕ that transforms as a Lorenz scalar. The situation is more complicated with fermions because of their spinor structure, but we use what we know about non-relativistic spinor rotations to elucidate the structure. We know how spin-1/2 states vary under ordinary spatial rotations

$$\psi \rightarrow e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\psi,$$

where the $\{\sigma_j\}$ are the usual Pauli matrices that generate SU(2), which acts on a space of states defined over the field **C**, to transmit the action of the group of rotations acting on Euclidean 3-space. So, a rotation in Euclidean configuration space shares a bijective correspondnece with rotations in state space($SU(2) \simeq O(3)$). We extend this idea to the full Lorenz group in state space, correcsponding to $SO(3,1) \simeq SO(4)$, of which the above spatial rotations are a subgroup. The Lorenz group acting on real 4-vectors is generated by the algebra $\{\{J_i\},\{K_i\}\}$, which has the following structure:

$$[J_i, J_j] = \epsilon_{ijk} J_k$$

$$[K_i, K_j] = -\epsilon_{ijk} J_k$$

$$[J_i, K_j] = \epsilon_{ijk} K_k.$$

Note the first set defines the independent subalgebra generating spatial rotations. We can combine these generators in a profitable way. Define

$$L_{\pm i} = \frac{1}{2}(J_i \pm iK_i).$$

This is an equivalent way of writing the generators of the Lorenz group, but instead of having the spatial rotations as a seperate group, we now have

$$[L_{\pm i}, L_{\pm j}] = i\epsilon_{ijk} L_{\pm k}$$

and most importantly

$$[L_{+i}, L_{-i}] = 0.$$

So the $L_\pm$ each satisfy the defining condition for $su(2)$. In other words, the Lorenz group acting on a complex vector space is reducible to two copies of $SU(2)$, or $SO(3,1) \simeq SU(2)\times SU(2)$, and we can write our states in terms of two-component spinors transforming as two independent representations. For the rest of the paper, we refer to these as Weyl spinors. We denote by χ^α the spinor transforming under L_+ (the *identity representation*), and $\bar{\chi}_{\dot{\alpha}}$ that transforming under L_- (the *conjugate representation*), using the dotted and undotted spinor, or *Van der Waerden* notation. In other words,

$$\chi_\alpha \to exp\{-i\vec{\sigma}\cdot\vec{\theta} + \vec{\sigma}\cdot\vec{\eta}\}\chi_\alpha$$

$$\bar{\chi}^{\dot{\alpha}} \to exp\{-i\vec{\sigma}\cdot\vec{\theta} - \vec{\sigma}\cdot\vec{\eta}\}\bar{\chi}^{\dot{\alpha}},$$

where $\vec{\theta}$ is our infinitessimal angle of spatial rotation, and $\vec{\eta}$ are the space time rotation angles(the boost angle or rapidity). If we were to perform a Lorenz boost on χ_α and $\bar{\chi}^{\dot{\alpha}}$, through a rapidity η and perform a rotation to get back the original state, that required angle θ would be equal and opposite for each of these cases. We boost along $\vec{p}$, the particle's momentum, which is the frame under consideration other than some lab frame, and the rotation back is generated by the particle's angular momentum. So, because of what we have just said, both particles have opposite values of $\vec{s}\cdot\vec{p}/|\vec{p}|$ ($\vec{s} = \pm\hbar/2$ is the particle's spin angular momentum). We say then that these particles have opposite *helicity.* $\bar{\chi}^{\dot{\alpha}}$ has helicity +1/2, and is called right-handed, and χ_α has helicity -1/2 and is called left-handed.

How do these transformations change under parity? Note above that reversing parity takes K_i to $-K_i$, so left or right-handed spinors by themselves transform into each other and are not parity eigenstates, meaning that physics only depending on one and not the other does not conserve parity. If we want physics that is invariant under parity, it must be in terms of states that combine the two to form a 4-component *Dirac spinor.*

$$\psi = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}.$$

The familiar Dirac equation for ψ is

$$(i\gamma^\mu\partial_\mu - m)\psi = 0,$$

where the γ^μ are 4×4 matrices that satisfy the relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

and are such that this equation, with our chosen representation of ψ is Lorenz invariant. This is the Weyl representation of the γ^μ, and is given by

$$\gamma^\mu = \begin{pmatrix} 0_{2\times 2} & \sigma^\mu \\ \bar{\sigma}^\mu & 0_{2\times 2} \end{pmatrix}.$$

$$(\sigma^\mu = (1_{2\times 2}, \vec{\sigma}), \bar{\sigma}^\mu = (1_{2\times 2}, -\vec{\sigma}))$$

From this equation it seems sensible that σ^μ carries both dotted and undotted indices-$\sigma^\mu_{\alpha\dot{\alpha}}$ and $\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}$, where we also have introduced our convention for raising and lowering indices using Levi-Civita symbols (direct computation shows this is consistent with our definition for $\bar{\sigma}$), and by convention, dotted indices are summed "northeast" and undotted ones are summed "southwest". The index structure of the Pauli matrices can be seen by considering the similarity transformations on them:

$$\sigma^\mu \longrightarrow M\sigma^\mu M^\dagger, M \in su(2).$$

If M is in the identity representation, then $M^\dagger$ is in the conjugate representation. For σ to transform this way, it needs a dotted and an undotted index. This representation can be shown to be equivalent(i.e., related by a similarity transformation) to Dirac's original prescription usually learned in single-particle relativistic quantum mechanics. The Dirac field equation comes from the Lagrangian

$$\pounds = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi,$$

where $\bar{\psi} = \psi^\dagger\gamma^0$, in order to ensure that $\bar{\psi}\psi$ transforms as a Lorenz scalar. In our second quantization procedure, we proceed analogously with scalar fields and write

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}}_s \{a^s_{\vec{p}} u^s(p) e^{-ipx} + b^{s\dagger}_{\vec{p}} v^s(p) e^{ipx}\}$$

$$\psi^\dagger(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}}_s \{a^{\dagger s}_{\vec{p}} u^{\dagger s}(p) e^{ipx} + b^s_{\vec{p}} v^{\dagger s}(p) e^{-ipx}\}$$

The sum over s is the sum over all possible spins. The 4-spinor behavior of fermions is absorbed into the u's, and that of antifermions into the v's, and this field operator serves to annihilate a fermion and to create an anti-fermion. Just as Klein-Gordon paricles' creation and annihilation operators were defined by commutation relations, those for fermions obey analogous relations, but in terms of anticommutators. This imposes the Pauli exclusion principle, (and ensures causality, it turns out; see Sec. 3.8 of [3]).

C. Internal Symmetries and Gauge Fields

In this section, we look at the interplay of vector fields (like the photon) with fields that constitute matter, as arising from internal symmetries in the problem. If a fermion field $\psi \to e^{i\alpha}\psi$ is changed by a constant phase, we get the same physics, i.e., the Lagrangian transforms covariantly under this type of transformation, leaving the field equations invariant:

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \to e^{ig\alpha}i\bar{\psi}\gamma^\mu\partial_\mu(e^{-ig\alpha}\psi) - m\bar{\psi}\psi,$$

but the story is different if α were *locally* dependent on space time coordinates, as the phase does not just move across the derivative. We have

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \to e^{ig\alpha}i\bar{\psi}\gamma^\mu(e^{-ig\alpha}(\partial_\mu - ig\partial_\mu\alpha)\psi - m\bar{\psi}\psi.$$

For this case, introducing the interaction of charged particles with an electromagnetic potential can impose symmetry under these local transformations.

$$\partial_\mu \to D_\mu = \partial_\mu - igA_\mu,$$

and since A_μ changes by some gradient under gauge transformations, we can say that under the transformation that multiplies ψ by a phase, we perform a corresponding gauge transformation on the potential. In the example given above, $A_\mu \to A_\mu + \partial_\mu\alpha$, and the Lagrangian is unchanged. This group of phase transformations is known as $U(1)$, and abelian unitary group,and is the simplest example of a gauge group. The A_μ are called gauge fields, and how many different such fields there are equals the number of generators in the group. $U(1)$ is abelian, therefore all of its irreducible representations are one-dimensional, so there is only one photon.

We can extend this idea to larger and more complicated groups of unitary transformations. We want to write the group elements in the form

$$E = e^{-ic_G G^i\alpha_i(x)},$$

with generators G^i, and coupling constant c_G. Such a continuous, infinitessimally-generated group is known as a Lie group. Their generators form what is called a Lie algebra, defined by some commutation relation

$$[G^a, G^b] = if^{abc}G^c.$$

The f^{abc} are called the structure constants of the group. We will talk more about this mathematical structure and its extension later when it would be more relevant. The important thing to note is this is how we describe and define symmetries in physics. An example of a nonabelian gauge group is $SU(2)$ (Special unitary group acting in 2 complex dimensions), the group of rotations in ordinary quantum mechanics, which is generated by the Pauli matrices:

$$T = e^{-i\vec{\sigma}\bullet\vec{\theta}/2}, [\frac{\sigma^a}{2}, \frac{\sigma^b}{2}] = i\epsilon^{abc}\frac{\sigma^c}{2}.$$

The general gauge covariant field strength tensor is given by

$$igG_a F^a_{\mu\nu} = [D_\mu, D_\nu],$$

the commutator of the covariant derivatives. It is clear that for $U(1)$, this gives the usual field strength tensor in electrodynamics. The fields in general transform as

$$A^a_\mu \to A^a_\mu + \partial_\mu \alpha^a - if^{abc}\alpha^b A^c_\mu.$$

We now have all of the background, at least enough to understand the language used in the paper.

III. THE SIMPLEST SUPERSYMMETRIC LAGRANGIAN

We want to put study how a fundamental symmetry between bosons and fermions comes about. To do this, we study the simplest Lagrangian that features both bosons and fermions. Our template Lagrangian is that of a non-interacting theory featuring a Weyl fermion, χ and a 2-component scalar :

$$L = -i\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi - \partial^\mu A^* \partial_\mu A.$$

Sometimes the theory we work with is known as the Wess-Zumino Lagrangian,[**?**], but that generally refers to a similar model defined with a single Majorana spinor and two scalar fields(see [9],[2],[7], and the way we are proceeding is done in [8],[6]; see also Problem 4, Chap.

3 of Peskin and Schroeder). We want to rotate a boson into a fermion infinitessimally, and we write this transformation

$$\delta_\epsilon A = \sqrt{2}\epsilon\chi,$$

where the $\sqrt{2}$arises from historical convention, and with benefit of hindsight. In order for spin statistics and mass dimensions to make sense, $\epsilon\chi = \epsilon^a \chi_a$ must behave as a scalar, meaning it must be defined through a commutation relation and have mass dimension 1. This requires that ϵ be a two-component Grassman number, defined by

$$\{\epsilon^a, \epsilon^b\} = 0,$$

and ϵ must have dimensions of $(mass)^{-1/2}$. This Lagrangian only changes by a total divergence if the fermion transforms in a way to cancel the extra terms generated by changing A. We can see that this is so if χ transforms as

$$\delta_\epsilon \chi_a = i\sqrt{2}(\sigma^\mu \bar{\epsilon})_a \partial_\mu A.$$

We look at the commutator of two successive transformations on A:

$$[\delta_\xi, \delta_\eta]A = 2i(\eta\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\eta})\partial_\mu A.$$

For χ, this commutator is

$$[\delta_\xi, \delta_\eta]\chi_a = 2i(\sigma^\mu\bar{\eta})_a \xi\partial_\mu\chi - i(\sigma^\mu\bar{\xi})_a \eta\partial_\mu\chi.$$

We apply the Fierz identity,

$$\chi_a(\xi\eta) = -\xi_a(\eta\chi) - \eta_a(\chi\xi),$$

for three spinors χ,η and ξ, to get

$$[\delta_\xi, \delta_\eta]\chi_a = 2(i(\eta\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\eta})\partial_\mu\chi - i\eta_a\bar{\xi}\bar{\sigma}^\mu\partial_\mu\chi + i\xi_a\bar{\eta}\bar{\sigma}^\mu\partial_\mu\chi),$$

the last two terms of which disappear if we are onshell, since the fermion was originally massless. So, onshell, we have the attractive relations

$$[\delta_\xi, \delta_\eta]X = -2i(\xi\sigma^\mu\bar{\eta} - \eta\sigma^\mu\bar{\xi})\partial_\mu X,$$

for $X = A, \chi_a$. If there is a fundamental symmetry in nature between bosons and fermions, then each should transform the same under a commutation of transformations since if this is the case, we have an algebraic relationship describing the infinitessimal generators of these transformations; if this is the case, we can write this symmetry as a group acting on a single field that takes A and χ as components. We want this symmetry to be true at every point in natural processes, which means that the commutator should be the same offshell, as well, which it is not at present. The way to solve this is to add what are known as auxilliary fields that make this commutator the same for the boson and fermion, but that vanish onshell. If we write a field F in the Lagrangian with no derivative terms or interactions with other fields,

$$\pounds_{auxilliary} = F^* F,$$

then F vanishes onshell, since its field equation is trivial. We want the theory to be supersymmetric for F also, so we want it to be transformed into χ. Note also that F has dimensions of $(mass)^2$. The most ovbious way to transform F and keep all indices consistent is to write

$$\delta_\epsilon F = i\sqrt{2}\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi.$$

This is cancelled if we augment the fermion's transformation to

$$\delta_\epsilon \chi_a = i\sqrt{2}(\sigma^\mu \bar{\epsilon})_a \partial_\mu A + \sqrt{2}\epsilon_a F.$$

If we want this symmetry to be a rotation between bosons and fermions, transforming A does not give rise to any F-fields. The algebraic relation above is also obeyed by F, and we have the commutation relation on supersymmetry operators

$$[\delta_\xi, \delta_\eta] = -2i(\xi\sigma^\mu\bar{\eta} - \eta\sigma^\mu\bar{\xi})\partial_\mu.$$

If we write this as some sort of unitary transformation on quantum states with generators Q_a and $\bar{Q}^{\dot{a}}$, then our transformations are represented by

$$(\epsilon^b Q_b + \bar{\epsilon}_{\dot{b}}\bar{Q}^{\dot{b}})A = \sqrt{2}\epsilon\chi$$

$$(\epsilon^b Q_b + \bar{\epsilon}_{\dot{b}}\bar{Q}^{\dot{b}})\chi_a = i\sqrt{2}(\sigma^\mu\bar{\epsilon})_a\partial_\mu A + \sqrt{2}\epsilon_a F$$
$$(\epsilon^b Q_b + \bar{\epsilon}_{\dot{b}}\bar{Q}^{\dot{b}})F = i\sqrt{2}\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi,$$

and our commutator expands to

$$[(\xi^b Q_b + \bar{\xi}_{\dot{b}}\bar{Q}^{\dot{b}}), (\eta^b Q_b + \bar{\eta}_{\dot{b}}\bar{Q}^{\dot{b}})] = \xi^b Q_b \bar{\eta}_{\dot{b}}\bar{Q}^{\dot{b}} + \bar{\xi}_{\dot{b}}\bar{Q}^{\dot{b}}\eta^b Q_b - \bar{\eta}_{\dot{b}}\bar{Q}^{\dot{b}}\xi^b Q_b - \eta^b Q_b \bar{\xi}_{\dot{b}}\bar{Q}^{\dot{b}},$$

making our fundamental relation

$$\xi^b Q_b \bar{Q}_{\dot{b}} \bar{\eta}^{\dot{b}} - \eta^b \bar{Q}_{\dot{b}} Q_b \bar{\xi}^{\dot{b}} + \xi^b \bar{Q}_{\dot{b}} Q_b \bar{\eta}^{\dot{b}} - \eta^b Q_b \bar{Q}_{\dot{b}} \bar{\xi}^{\dot{b}} = -2i(\xi^b \sigma^\mu_{b\dot{b}} \bar{\eta}^{\dot{b}} - \eta^b \sigma^\mu_{b\dot{b}} \bar{\xi}^{\dot{b}})\partial_\mu$$

Comparing terms with the same Grassman parameters gives us the fundamental algebraic relations defining for us how supersymmetry works:

$$\{Q_a, Q_b\} = \{\bar{Q}_{\dot{a}}, \bar{Q}_{\dot{b}}\} = 0$$

$$\{Q_b, \bar{Q}_{\dot{b}}\} = 2\sigma^\mu_{b\dot{b}} P_\mu,$$

where $P_\mu = -i\partial_\mu$, the 4-momentum operator, is the generator of space-time translations.

IV. THE SUPERSYMMETRY ALGEBRA

In the last section, we found from the simplest possible supersymmetric Lagrangian a defining algebraic relation between the generators of SUSY, Q_a and $\bar{Q}_{\dot{a}}$now.

$$\{Q_a, \bar{Q}_{\dot{a}}\} = 2\sigma^\mu_{a\dot{a}} P_\mu.$$

When we add P_μ to our list of spacetime symmetry group generators, we have all possible rotations and translations in spacetime, all related through commutation relations, with

$$[P_\mu, P_\nu] = 0$$

$$[M_{\mu\nu}, P_\lambda] = i(\eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}),$$

where we have used the notation $M_{\mu\nu} = \epsilon_{\mu\nu\lambda} J^\lambda$. From our experience with quantum systems, transformations of internal space are handled independently of the space-time symmetries. We can transform states in non-relativistic quantum mechanics without fear of having transformed to another inertial reference frame, for example. This is actually for a good reason, as symmetry groups are generally generated by a Lie algebra. Coleman and Mandula[12] proved a theorem where, based on general assumtions, if we try to write down a complete group of symmetries for a scattering matrix, one can only combine the Poincare group trivially with whatever group of internal symmetry groups transform the states. In other words, these two kinds of symmetries can be at most combined as a direct sum; recall how $SU(3)$

color symmetry has nothing whatever to do with relativity or translational invariance. More officially, the Coleman-Mandula theorem states that the most general Lie algebra of symmetries of the S-matrix contains the Poincare group, and a finite number of operators B_j that transform as Lorenz scalars ($[M_{\mu\nu}, B_j] = [P_\mu, B_j] = 0$), with the B's forming their own Lie algebra, defined by

$$[B_i, B_j] = iS_{ij}^k B_k,$$

some structure constants S_{ij}^k. Now, it seems as though the supersymmetry algebra violates this; it is a transformation in field-space, but also give rise to space-time translation. This works with the Coleman-Mandula theorem because all symmetries they considered were generated by Lie algebras, which are defined by their closed *commutator* relations.

A Lie Algebra L_0 is defined by a product

$$\bullet : L_0 \times L_0 \to L_0,$$

where

$$v_1 \bullet v_2 = v_1 v_2 - v_2 v_1,$$

and

$$v_1 \bullet (v_2 \bullet v_3) + v_2 \bullet (v_3 \bullet v_1) + v_3 \bullet (v_1 \bullet v_2) = 0$$

(Jacobi identity),$\forall v_1, v_2 \in L_0$. What supersymmetry does is it puts an extra structure on our physical symmetries, and we have anticommutator as well as commutator relations defining the structure. We say that supersymmetry puts a *grading* on our Poincare group. We define a *graded Lie algebra* as a direct sum

$$L = L_0 \oplus L_1 \oplus ... \oplus L_J,$$

with a product on L, $\bullet$ (this is what we call the "grading")defined so that

$$\bullet : L_i \times L_j \to L_{(i+j) mod(N+1)},$$

such that

$$x_m \bullet x_n = -(-1)^{g(x_m)\times g(x_n)} x_n \bullet x_m,$$

for $x_j \in L_{g(x_j)}$, and we have the generalization of the Jacobi identity

$$(-1)^{g(x_i)g(x_k)} x_i \bullet (x_j \bullet x_k) + (-1)^{g(x_j)g(x_i)} x_j \bullet (x_k \bullet x_i) + (-1)^{g(x_k)g(x_j)} x_k \bullet (x_i \bullet x_j) = 0.$$

We see that if we start with both x_m and x_n in L_0, then the product between them is a commutator, and is taken back to L_0, as it should be as it spans the ordinary Lie algebra. For $g(x_m) = 0, g(x_n) = 1$, the product between the two is also a commutator and the result is in L_1. Given a Lie algebra, we can find its grading dependent on the dimension using the generalized Jacobi identity. For example, we can choose to find the $\mathbf{Z}_2$-grading on $su(2)$. In the matrix representation acting in two dimensions, L_1 consists of the Pauli matrices. In a three-dimensional representation, it consists of the Levi-Civita symbols. Note that, although L_0 is always a subalgebra, the other L_j are not in general, since their products are not closed. Note also that the degree of the grading is isomorphic to $\mathbf{Z}_{J+1}$, and since the supersymmetry algebra contains the commuting and anticommuting parts, we have $L_{SUSY} = L_0 \oplus L_1$, so we say that supersymmetry is a $\mathbf{Z}_2$-graded Lie algebra.

We want our supersymmetry to be (for our purposes) global, giving its product with the P_μ:

$$[Q_a, P_\mu] = [\bar{Q}_{\dot{a}}, P_\mu] = 0.$$

How do our Q's products with the Lorenz group behave? By our results and definitions above, the product is a commutator, and the result lives in L_1, and it must be nonzero in general; since the Q's commute with P_μ, they can only simultaneously commute with the μ-th component of angular momentum. We can use the index structure to write

$$[M_{\mu\nu}, Q_\alpha] = (S_{\mu\nu})_\alpha^\gamma Q_\gamma,$$

and see from the generalized Jacobi relation (take Q and two of the M's) that the $(S_{\mu\nu})_\alpha^\gamma$ satisfy the algebra that defines the Lorenz algebra, and so are given by a representation of it. We work in the two-component Weyl representation of SUSY, so our representation must be 2-dimensional, so built out of Pauli matrices. With the index structure of the spinor indices, we conclude that

$$(S^{\mu\nu})_\alpha^\gamma = -\frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\nu).$$

Consider the special case of angular momentum along the z-axis $J_3 (= M_{12})$:

$$[M_{12}, Q_1] = -\frac{1}{2} Q_1$$

$$[M_{12}, Q_2] = \frac{1}{2} Q_2,$$

putting through the commutation relations among the Pauli matrices. Likewise,

$$[M_{12}, \bar{Q}_{\dot{1}}] = \frac{1}{2}\bar{Q}_{\dot{1}}.$$

$$[M_{12}, \bar{Q}_{\dot{2}}] = -\frac{1}{2}\bar{Q}_{\dot{2}}.$$

From this, we see that $Q, \bar{Q}$ have the same commutation relations for raising and lowering (depending on the index) as the angular momentum raising and lowering operators in ordinary quantum mechanics, except there is the factor if 1/2, so the angular momentum of a state is lowered in increments of 1/2 instead of 1. Because of the commutation with linear momentum, we see that these transformations act on helicity states $|\lambda\rangle$ s.t. $(\vec{p} \bullet \vec{\sigma}/|\vec{p}|)|\lambda\rangle = \lambda|\lambda\rangle$ in order to raise and lower the helicities of particles in increments of 1/2, thereby actually transforming bosons into fermions and vice-versa.

We could extend the SUSY algebra we have been considering to a larger dimension, defining the fundamantal relation

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\delta^{IJ}\sigma^\mu_{\alpha\dot{\alpha}}P_\mu$$

$$I, J = 1, ..N.$$

Such an affair is called *extended supersymmetry*, and are denoted by N (the physically-important representation of the SUSY algebra we have been considering is known simply as "N = 1 supersymmetry"). The Kronecker delta is always allowed; whatever matrix is on the right-hand side needs to be hermition and positive-definite, and therefore there exists some similarity transformation that diagonalizes it, which we absorb into the generators and define them as above. We consider N = something SUSY, with state $|\lambda\rangle$ the state of highest helicity. Our Q^I_1 acts as lowering operators (ignoring normalization factors), and so acts like

$$Q^{I_1}_1|\lambda\rangle = [I_1]|\lambda - \frac{1}{2}\rangle$$

$$Q^{I_2}_1 Q^{I_1}_1|\lambda\rangle = [I_1 I_2]|\lambda - 1\rangle$$

$$...$$

$$Q^{I_N}...Q^{I_1}|\lambda\rangle = [I_1...I_N]|\lambda - N/2\rangle,$$

where the bracketed symbols are completely antisymmetric in the indices, from the anti-commutation relations, and so there can only be at most N operations on a state. In order for the universe to be CP-symmetric, we must be able to act on $|-\lambda\rangle$ and get the same particle spectrum, but with opposite helicities. The maximum possible N that will allow this is $N/2 = 2\lambda$ (i.e., the situation where the generators take $|\lambda\rangle$ clear all the way through to $|-\lambda\rangle$). Examples of such a *maximal* extension are $N = 4$ Yang-Mills theory($\lambda = 1$) and $N = 8$ supergravity ($\lambda = 2$), owing to the spin-1 of gauge fields and the hypothetical spin-2 of the graviton.

V. SUPERFIELDS AND SUPERSPACE

If supersymmetry is a fundamental symmetry of nature, it should act like one. We should have our SUSY generators acting on a field and performing internal rotations to turn fields of one statistical behavior into the other. From this idea we get the notion of a *superfield*, $\Phi(x, \theta, \bar{\theta})$, which has bosons and their fermionic partners as components. We define Φ to act as a scalar, and expand it in the $(mass)^{1/2}$-dimension Grassman parameters to have the fermion components behave like this. The elements of the group of SUSY transformations we would think should be written

$$F(\theta, \bar{\theta}) = e^{i(\theta Q + \bar{\theta}\bar{Q})},$$

except that we know SUSY is special in that space-time translations are generated by it, so to get group closure (regard the Baker-Hausdorff lemma), we must include spacetime translations also, so we write

$$G(x, \theta, \bar{\theta}) = e^{i(-x^\mu P_\mu + \theta Q + \bar{\theta}\bar{Q})}$$

as our supersymmetry transformation. A motivating picture of the space that our fields move around in is that its actual spacetime consists of the coordinates $(x^\mu, \theta, \bar{\theta})$, consisting of 4 bosonic and 4 fermionic coordinates. This generalization of spacetime is known as *superspace*.

Let us take our superfield, and act on it with a pure supersymmetry transformation:

$$\Phi(x, \theta, \bar{\theta}) \to G(0, \xi, \bar{\xi})\Phi(x, \theta, \bar{\theta}),$$

and take our original coordinates close to the origin, and write it as a field removed infinitesimally from the origin in superspace

$$\Phi(x, \theta, \bar{\theta}) = G(x, \theta, \bar{\theta})\Phi(0, 0, 0),$$

$$\Phi(x,\theta,\bar\theta) \to G(0,\xi,\bar\xi)G(x,\theta,\bar\theta)\Phi(0,0,0)$$

$$G(0,\xi,\bar\xi)G(x,\theta,\bar\theta) = e^{i(\xi Q+\bar\xi\bar Q)}e^{i(-x^\mu P_\mu+\theta Q+\bar\theta\bar Q)}.$$

We recall our (anti-)commutation conditions, and apply Baker-Hausdorff to get

$$G(0,\xi,\bar\xi)G(x,\theta,\bar\theta) = e^{i(-(x^\mu+i\theta\sigma^\mu\bar\xi-i\xi\sigma^\mu\bar\theta)P_\mu+(\theta+\xi)Q+(\bar\theta+\bar\xi)\bar Q)},$$

thereby giving us our superspace connection coefficients

$$\delta x^\mu = i\theta\sigma^\mu d\bar\theta - id\theta\sigma^\mu\bar\theta$$

$$\delta\theta = \delta\bar\theta = 0,$$

where $d\theta$ is the infinitessimal element we called ξ above. In addition, we can now define $Q, \bar Q$ as differential operators in superspace

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar\theta_\alpha)\partial_\mu$$

$$\bar Q_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu.$$

Now, if we write our superfield as an expansion in $\theta, \bar\theta$, we know exactly how it transforms under supersymmetry.

If what we are talking about is a symmetry that nature actually follows, we must consider actions that are invariant under supersymmetry. An action in our superuniverse must be given as an integral over *all* coordinates, even the Grassman coordinates. Because θ and $\bar\theta$ each have 2 components, a general action over superspace is of the form

$$S = \int d^2\theta d^2\bar\theta H(\theta,\bar\theta) + \int d^2\theta I(\theta) + \int d^2\bar\theta\bar I(\bar\theta).$$

It is well-known how integrals over Grassman parameters behave:

$$\int d\theta = 0; \int \theta d\theta = 1,$$

along with derivatives:

$$\frac{\partial}{\partial\theta^a}(\theta^b\theta^c) = \delta^{ab}\theta^c - \theta^b\delta^{ac}.$$

It follows that

$$\int \theta\theta d^2\theta = \int \bar\theta\bar\theta d^2\bar\theta = 1.$$

The only terms in the functions F and I above that contribute to the action therefore are those that go like $\theta\theta\bar{\theta}\bar{\theta}$ and $\theta\theta$, respectively. Because these are two-component, this is the highest power of the parameters that is nonzero. Therefore, from the form of the supersymmetry generators as differential operators, the relevant terms in the lagrangian transform as a total derivative under supersymmetry. This means that any action we construct from superfields in this superspace is automatically invariant under supersymmetry.

Also worthy of remark are the differential operators

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu.$$

These anticommute with the supersymmetry generators; therefore they transform superfields into superfields.

VI. CHIRAL SUPERFIELDS AND WESS-ZUMINO

In an earlier section, we discussed a supersymmetric model consisting of a scalar, a fermionic partner, and a non-propagating auxiliary scalar. How do we recover this using superfields? The model is an example of what is known as a chiral superfield,$\Phi(x,\theta,\bar{\theta})$ defined by the condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0.$$

If we write our superfield as an expansion in θ, as a function of $y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$, then in these new coordinates, $\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}$. We write $\Phi(y,\theta)$ out as such an expansion.

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y).$$

Expanding gives (using well-known results from spinor algebra)

$$\Phi(x,\theta,\bar{\theta}) = A(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu A(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2 A(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta\theta F(x)$$

Furthermore,

$$\Phi^\dagger(x,\theta,\bar{\theta}) = A^*(x) - i\theta\bar{\sigma}^\mu\bar{\theta}\partial_\mu A^*(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2 A^*(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) + \bar{\theta}\bar{\theta}F^*(x),$$

and multiplying these fields together gives us a function of $\theta, \bar{\theta}$, and we only retain terms that go as $\theta\theta\bar{\theta}\bar{\theta}$:

$$\Phi^\dagger\Phi|_{\theta\theta\bar{\theta}\bar{\theta}} = F^*F + \frac{1}{4}A^*\partial^2 A + \frac{1}{4}\partial^2 A^* A - \frac{1}{2}\partial_\mu A^*\partial^\mu A + \frac{i}{2}\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi.$$

Through integration by parts, we see that this gives the kinetic terms of the same action as the Wess-Zumino model we considered before. Although we have only one field in each component, we could give them internal indices and write $\Phi_i^\dagger\Phi_j$. So, we have here a combination of superfields that give us back our ordinary kinetic terms in a lagrangian, so this is what we write for the kinetic terms of scalars/fermions and their suerpartners.

We now construct how superfields interact, in a way that gives us back our usual particle interactions. The mose general potential made of chiral superfields is the third-degree polymonial

$$W(\Phi) = \lambda^i\Phi_i + \frac{1}{2}m^{ij}\Phi_i\Phi_j + \frac{1}{3}y^{ijk}\Phi_i\Phi_j\Phi_k,$$

called the *superpotential*, which is itself a chiral superfield. W can me at most third degree because the integral over the Grassman parameters contributes a factor of mass dimension one. If we write our fields in terms of the coordinate y, then the superpotential is only dependent on θ, and so only the terms that go as $\theta\theta$ contribute to the action. That is,

$$\Phi_i\Phi_j|_{\theta\theta} = A_i(y)F_j(y) + A_j(y)F_i(y) - \psi_i(y)\psi_j(y)$$

$$\Phi_i\Phi_j\Phi_k|_{\theta\theta} = F_iA_jA_k - \psi_i\psi_jA_k + cyc.perm.ijk.$$

The interesting thing to note is that when we include these interactions, the field equation for F is no longer trivial, and we can find the equation and substitute for F. Back to the simple case of one field,

$$\partial_F \pounds = 0 = F^* - mA - \frac{1}{2}yAA$$
$$\partial_{F^*} \pounds = 0 = F - mA^* - \frac{1}{2}yA^*A^*,$$

So we see from above that we recover the usual quartic self-interaction that is needed for renormalizability, as well as the scalar mass and the Yukawa interaction between the scalars and fermions in the theory. The actual potential energy in the lagrangian must be Hermitian, so we add to W its Hermitian conjugate, and in doing so, we recover the chiral symmetry-violating fermion masses, as we will see later.

VII. GAUGE SYMMETRY AND VECTOR SUPERFIELDS

In the introduction, we stressed that any theory should be invariant under some group of local phases. We examine abelian $U(1)$ transformations. If nature is supersymmetric, these phases should act on a superfield (in other words, all of the components should be rotated by the same phase), and, knowing what we know, we would expect to add a field that transforms as a vector that transforms in such a way to impose symmetry under these transformations. Here, the discussion is almost entirely that of Chapters 6-7 of [1]. Write these transformations as

$$\Phi \to e^{-i\Lambda}\Phi$$
$$\Phi^\dagger \to \Phi^\dagger e^{i\Lambda^\dagger},$$

where Λ is a chiral superfield.

To ensure that our kinetic energy is gauge invariant, we augment our definition by writing our transformed kinetic energy as

$$\Phi^\dagger e^{gV}\Phi \to \Phi^\dagger e^{i\Lambda^\dagger} e^{gV'} e^{-i\Lambda}\Phi.$$

Clearly, we require that $V = V^\dagger$. We could write an absolutely general expansion of V in powers of $\theta, \bar{\theta}$, with many different component fields, but as this is $N = 1$ supersymmetry, most of them would be unphysical, and can be gotten rid of by some choice of Λ that does not affect anything physically. Picking such a Λ is known as the *Wess Zumino gauge.* (For further discussion, consult Wess and Bagger, Ch. 6,7.) The result left over after choosing the WZ gauge is

$$V = \theta\sigma^\mu\bar{\theta}v_\mu + i\theta\lambda - i\bar{\theta}\bar{\lambda} + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D.$$
$$\delta V = i\theta\sigma^\mu\bar{\theta}\partial_\mu(A - A^*).$$

These are the only fields that are not canceled by some possible chiral superfield $\Lambda, \bar{\Lambda}$. The vector superfeld is called so because of the vector field (the photon) v_μ, that transforms under further gauge transformations just as usual, and its superpartners $\lambda, \bar{\lambda}$, fermions known as gauginos, and an auxiliary scalar D, which, like the F has no derivative terms in the Lagrangian (we know that since there cannot be any more powers of $\theta, \bar{\theta}$ in the terms with D). The analogue of the field strength tensor is given by

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V,$$

with D's the differential operators defined above (and totally unrelated to the auxiliary scalar field D). This generalizes to non-abeilian gauge theories, with $\Lambda_{ij} = T^a_{ij}\Lambda_a, V = T^a V_a$, for gauge group generators T^a. The SUSY extension of the field strength tensor is generalized to

$$W_\alpha = \bar{D}\bar{D}e^{-V}D_\alpha e^{eV}.$$

We now have everything we need to make supersymmetric generalizations of field theories.

VIII. AN APPLICATION: SUSY QED

Quantum electrodynamics(QED) is the simplest field theory of physical relevance, and the best-tested theory in the history of mankind. It is only sensible that we start here in talking about supersymmetric applications to physical systems. The lagrangian that gives rise to QED is the usual Dirac Lagrangian, augmented with the vector potential to ensure the $U(1)$ gauge symmetry.

$$\pounds_{QED} = i\bar{\psi}_D\gamma^\mu(\partial_\mu - ieA_\mu)\psi_D - \bar{\psi}_D m\psi_D.$$

The ψ_D's here are general 4-component Dirac fermions, the electron, with left and right-handed parts.

$$\psi_D = \begin{pmatrix} \psi_{1\alpha} \\ \bar{\psi}_2^{\dot{\alpha}} \end{pmatrix}.$$

We supersymmetrize this theory by considering the electron as two Weyl fermions put together, and these as the fermionic components of two chiral superfields, and the photon vector potential as the vector component of a vector superfield, and write the kinetic part of the Lagrangian just as above. If we use the potential

$$P(\Phi) = m(\Phi_1\Phi_2 + \Phi_1^\dagger\Phi_2^\dagger),$$

the Lagrangian becomes

$$\pounds_{SUSYQED} = \Phi_1^\dagger e^{eV}\Phi_1 + \Phi_2^\dagger e^{-eV}\Phi_2 + m(\Phi_1\Phi_2 + \Phi_1^\dagger\Phi_2^\dagger) + \frac{1}{4}(WW + \bar{W}\bar{W}).$$

This is identical to the lagrangians we have been considering, except for the mass term. The only terms that contribute to the action are those that go like $\theta\theta$ and $\bar{\theta}\bar{\theta}$. From the general expansion, we know that this is, in terms of component fields,

$$P(\psi_j, A_j, F_j) = m(\psi_1\psi_2 + \bar{\psi}_1\bar{\psi}_2 + A_1F_2 + A_2F_1 + A_1^*F_2^* + A_2^*F_1^*).$$

If we define the Dirac fermion exactly as above, the fermionic part of this is simply $\psi_D^\dagger\gamma^0 m\psi_D$, which is what we wanted.

IX. AN APPLICATION: THE MINIMAL SUPERSYMMETRIC STANDARD MODEL

Of course, we need to build some sort of idea of how supersymmetry looks in particle physics. It is most sensible to see how the Standard model is minimally extened, and what that predicts. This is only a brief discussion of how the usual particles fit into the framework we have been discussing, and I refer the reader to Chapters 1 and 5 of Martin,[6].

A. The Standard Model of Particle Physics

Just as $U(1)$ symmetry defines QED, the particle content of the Standard Model is defined by those fields that transform as vectors under $SU(3)_C \times SU(2)_L \times U(1)_Y$. The vectors that transform as **3** under $SU(3)$ carry what is known as color charge(C) and are the *quarks*; the vector boson that goes with this symmetry is the *gluon*, of which there are 8, one for each generator. The particles that transform non-trivially under $SU(2)_L$ are all left-handed fermions. Left-handed quarks and leptons transform as two-component vectors consisting of a pair of particles with different *isospin* quantum numbers. For instance, the electron and its neutrino transform like

$$l_L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$$

$$l_R = (e_R)$$

(There are no right-handed neutrinos in the Standard Model.) Left-handed quarks (the up and down are isospin partners, as are charm and strange and bottom and top)form this type of vector, with the additional 3-component vector glued on. A particle and its isospin partner have electric charges that differ by 1 unit of e. $SU(2)$ has three generators, and so there are three vector bosons that come with it, two oppositely charged and one neutral, the W^+ ,W^- , and Z^0. The Noether currents that go with each of these fields are $j_+^\mu = \bar{e}_L^+ \gamma^\mu \nu_L$, $j_-^\mu = \bar{\nu}_L \gamma^\mu e_L^-$, and $j_0^\mu = \bar{e}^+ \gamma^\mu e^-$, respectively. The first two are what is observed in muon decay or nuclear beta decay, for example. The current that goes with the neutral Z^0 is the same as the conserved current that goes with the photon. This means that the two forces can be mixed together in building the universe we observe, and the photon and the Z are linear combinations of the original gauge fields(or they make up a 2-component vector that is a vector of the original fields, rotated by what is known as the Weinberg angle). We call these original neutral fields in $SU(2) \times U(1)$ W_μ^3 and B_μ. The $U(1)$ transformations are just multiplications by a phase. In order to preserve the $SU(2)$ symmetry, both components of those doublets need to transform by the same amount under one of these multiplications. In other words, the $U(1)$ charge for both elements of one of those vectors is the same for both of them, and is called *hypercharge*, Y, and has the following definition in terms of electric charge and isospin quantum number:

$$Q = I_3 + Y/2.$$

(The Gell-mann-Nishijima formula.)

The most interesting part about the $SU(2)$ portion of the standard model is that it must be broken in the universe in the condition we observe it, since the W's and Z have masses around 80 and 90 GeV, respectively, which actually make them some of the heaviest observed particles. It turns out that if these particles have masses, the action is no longer $SU(2)$ invariant. How the standard model explains this is by positing the existance of a new particle, a scalar whose ground state (or 'vacuum') has a nonzero amplitude (vacuum expectation value, or vev). If this particle transforms as **2** under $SU(2)$, then, if its vev is v, from the form we discussed of covariant derivatives and kinetic energies in the kinetic terms in scalars' Lagrangians, we van expand this scalar around its vev and find the W's will acquire a mass

$$m_W^2 = 2g_W^2 v^2,$$

which is what the coefficient of W^2 is the Lagrangian becomes. Such is the Higgs mechanism, and the scalar with nontrivial vev is known as the Higgs boson. So, near the ground state of the universe, we have a mass for these vector bosons, and the $SU(2)$ symmetry is said to be broken spontaneously. In this way, $SU(2)_L \times U(1)_Y$ is busted down to the $U(1)$ of electrodynamics, and (W^3_μ, B_μ) are rotated into (Z^0_μ, A_μ) in a combination chosen by nature in doing the breaking. Because photons are massless, the Higgs does not couple to them, and although it must have a charged component (in a doublet with different isospins and the same hypercharge), the vacuum is electrically neutral. Fermions get their masses from Higgs-fermion-fermion couplings in the potential. The Higgs boson is the only unobserved part of the Standard Model, and is the only scalar particle contained in it. Trying to verify this strange but elegant solution to the problem is presently very big business, indeed.

B. The MSSM Lagrangian

With superfields, contructing the supersymmetric extension of the Standard Model is almost done for us. We take all observed particles, and upgrade them to being components of their own superfields, and have the theory invariant under $SU(3)_C \times SU(2)_L \times U(1)_Y$. Standard Model fermions are part of chiral superfields, and the gauge bosons are vector components of vector superfields. We write our superfields in terms of the $SU(2)$ doublets, i.e., $U_L = (u, d)$. The left and right-handed quark, lepton and Higgs superfields for the first generation of particles is

$$\Phi^T = (U_L, E_L, u_R, d_R, e_R, H_u, H_d)$$

and the Higgs Yukawa couplings and hence particle masses are gotten from the superpotential

$$P(\Phi) = \mu H_u H_d + g_u \bar{u}_R H_u U_L + g_d \bar{d}_R H_d U_L + g_e \bar{e}_R H_d E_L,$$

with an identical pattern for the two higher generations (even SUSY has no answer for why there should only be three generations). The kinetic part of the Lagrangian is, as usual

$$\Phi^\dagger \exp\{\frac{g_C}{2}\lambda_i G^i + \frac{g_L}{2}\sigma_a W^a + YB\}\Phi.$$

(In our lazy notation, the g's represent matrices of the appropriate charges, and also the λ's and σ's are the usual Gell-mann and Pauli matrices that generate $SU(3)$ and $SU(2)$.)

The only thing that might strike the reader as strange is that there are now *two* Higgs bosons in the theory. There are (at least) two reasons why we need more than one. We want the superpotential to by a chiral superfield, which it would not be if we left it in the

FIG. 1: The quark contributions to the triangle diagrams that produce nonconserved current. This is a contribution with any fermion that interacts weakly, as the Higgsinos do.

$\mu H^\dagger H$ we might expect at first. The other, less artificial, reason is that when we have the Higgs' fermion superpartners (the Higgsinos), they interact with the weak gauge fields like any fermion. In a theory with only quarks and no leptons, there exist triangle graphs that consist of a W^+, W^- and a Z that are linearly divergent and give rise to a non-conserved current (the Bell-Jackiw anomaly, see Ryder, Ch. 5). Fortunately, these lepton diagrams' current is cancelled by those for the leptons. With these two Higgses of opposite charge, this cancellation happens among the same problem with Higgsinos.

Recalling the terms in the superpotential that contribute to the action, we see that we retrieve all of our Higgs-fermion-fermion interactions, and all of our interactions of fermions with gauge fields. The scalar partners of quarks and leptons are known as squarks and sleptons. Our interactions in the potential must go as $\theta\theta$ or $\bar{\theta}\bar{\theta}$, so in an interaction, the number of undiscovered squarks or higgsinos is $0 \mod 2$. What we have written is just the usual SM interactions, but with superfields instead of the original components. If our only concern were that the Lagrangian is supersymmetric, recall from earlier, that any combination of chiral superfields would do. We could add unobserved and baryon number-violating terms such as

$$\Delta P = \frac{1}{2} g^{ijk} E_{L,i} E_{l,j} \bar{e}_{R,k} + h^{ijk} E_{L,i} U_{L,j} \bar{d}_{R,k} + \alpha^i L_i H_u + \frac{1}{2} n^{ijk} \bar{u}_{R,i} \bar{d}_{R,j} \bar{d}_{R,k}.$$

The last term violates baryon number conservation by 1, and leads to proton decay, which is not observed.

This addition is not present in the Standard Model, but is there some underlying symmetry that these terms violate? There is, one known as R-parity, which is defined by

$$RP = (-1)^{3(B-L)+2s},$$

where B and L refer to baryon and lepton number, respectively. This transformation does not commute with SUSY. The Higgs and all observed Standard Model particles all have R-parity 1, and all of the squarks, sleptons, gauginos, and Higgsinos all have R-parity -1. This means that in order to have a Lagrangian that is unchanged under R-Parity, we actually do need an even number of as yet unobserved particles (supersymmetric particles or sparticles) in each interaction term. This has the consequence that the lightest supersymmetric particle (LSP) does not decay, as that would violate energy conservation. This means that there is a light particle that is perfectly stable and we have not seen it. It is thought to be a combination of the neutral parts of the Higgsino and the neutral Wino (SUSY partners of the W's) known as a neutralino, and if this is the case, then Supersymmetry with R-parity produces a dark matter candidate.

C. Why Supersymmetry is Important

Aside from its aesthetic value, there are practical problems that SUSY answers, and that is why it is at the top of the list of things to find experimentally. The Higgs and all scalar particles have the problem that if we try to compute the radiative corrections to their amplitudes in loop diagrams, we find corrections that are quadratically divergent. Fermion and vector boson masses are protected by symmetry already in the theory, and their radiative corrections are only logarithmically divergent, which we have methods of absorbing cleanly into the original mass to create a well-behaved mass correction. No such symmetry exists when this is done for scalars. There is no way to get rid of that infinity, and the Higgs mass must depend on an energy scale higher than we can do experiments in or even talk about physical theories. This is what is called the hierarchy problem. The lion's share of this problem comes from the loops with virtual top quarks, as that is the most massive known fermion. In the MSSM, this problem is solved by the top squark (or stop), which has quartic couplings to the Higgs, and the same diagram with a stop loop cancels the original that gave us our troublesome dependence on higher energy scales. Of course, this can be complicated by the fact that if supersymmetric particles had the same mass as particles we have seen, SUSY would have been observed experimentally, so it must be broken. However, there exist models (see discussion in [6, 9]) where SUSY is spontaneously broken in a way that it still solves the hierarchy problem, and you can do this by only correcting the squark and slepton masses and those of the gauginos. In that instance, SUSY is said to be broken *softly*. Spontaneous supersymmetry breaking is a fascinating topic, but there is no space for it here, and it was not mentioned in the original talk.

Also important is that SUSY solves a problem in grand unification. We would like, at some energy scale, to unify the three SM forces. What this means it we want to embed

FIG. 2: Problematic radiative correction to the Higgs mass from the top quark, and the correction from the stop that cancels it.

FIG. 3: In the usual Standard Model, the coupling constants of the fundamental forces do not all unify at any scale, but they do in the MSSM. Image courtesy of Gustaaf Brooijmans, ucl.edu.

$SU(3)$, $SU(2)$, and $U(1)$ into one big gauge group (the first such attempt was $SU(5)$). See Zee's text for an excellent introduction to grand unification. In order for this to happen, the gauge fields must couple to a particle each with the same strength in order to act as one element from a single group. If we plot the dependence of the coupling constants as a function of the scale we use, the three charges in the Standard Model do not all unify anywhere. However, in the MSSM, they actually do unify. This is one of the most exciting aspects of supersymmetry's predictions.

X. CLOSING WORDS AND OUTLOOK

The focus of this talk was on presenting in as consise and yet complete a way as space allows, what supersymmetry is and where it comes from. I tried to develop the idea of a symmetry between bosons and fermions intuitively, and then we saw that it comes from a very solid and established mathematical structure, which gives it an aesthetic appeal. We looked at supersymmetric lagrangians, and saw the language in which SUSY is presented, and found some applications in extensions of well-known models. I focused on the aesthetic and structural elegance of SUSY, but also touched on its relevance to all of particle physics.

Although supersymmetry remains completely experimentally unverified, it is of the utmost relevance to current phenomenological work and future experimental searches. This was conceived originally for string theory in 1+1 dimensions, and now it is among the most important new physics to be sought out at the LHC. This is much like gauge theories and the Higgs mechanism, which were researched for their mathematical beauty, and only later formed the basis of the Standard Model of particle physics. It seems impossible that nature would not make use of a perfectly good symmetry that comes from such basic mathematical consideration and an extension of known symmetries. Physics grows and thrives because of experiment, though. That must have the last word. We live in a universe where nature does not make use of a basic symmetry between right and left, and different time intervals are measured by two different inertial reference frames, but that is only to preserve the length of a line in actual physical space. Supersymmetry is very beautiful and convenient, but we are on the verge of actually putting these ideas to the test, and we are due for nature to once again surprise us all.

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